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PERFECT TOTIENT GROUPS

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ABSTRACT. Let G be a finite group and $\varphi(G) = |\{a \in G \mid o(a) = \exp(G)\}|$, where $o(a)$ denotes the order of a in G and $\exp(G)$ denotes the exponent of G. We say that G is a *perfect totient group* (or a *PT-group*, in short) if $|G| = \sum_{i=1}^{c_G} \varphi^i(G)$, where $c_G = \min\{m \in \mathbb{N}^* \mid \varphi^m(G) = 1\}$. In this note, several results concerning PT-groups are presented.

1. INTRODUCTION

The *Euler's totient function* (or, simply, the *totient function*) φ is one of the most famous functions in number theory. Notice that the totient $\varphi(n)$ of a positive integer n is defined to be the number of positive integers less than or equal to n that are coprime to n. The totient function is important mainly because it gives the order of the group of all units in the ring $(\mathbb{Z}_n, +, \cdot)$. Alternatively, $\varphi(n)$ can be seen as the number of generators, as the number of elements of order n or as the number of automorphisms of the finite cyclic group $(\mathbb{Z}_n, +)$.

Related to the totient function, we have the following arithmetical concept.

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Definition 1. *A positive integer n is called a perfect totient number (or a PT-number, in short) if it is equal to the sum of its iterated totients, that is*

$$n = \sum_{i=1}^{c_n} \varphi^i(n),$$

where $c_n = \min\{m \in \mathbb{N}^* \mid \varphi^m(n) = 1\}$.

Clearly, all PT-numbers are odd. It can be also observed that many PT-numbers are multiples of 3. In fact, 4375 is the smallest PT-number that is not divisible by 3. Several classes of PT-numbers are known. One of them consists of all powers of 3. Another one is given by the following criterion: if $p = 4 \cdot 3^k + 1$ is a prime, then $3p$ is a PT-number. Moreover, for an odd prime p we have that $3p$ is a PT-number if and only if $p = 4q + 1$ with q a PT-number. We note that one of the most important problem with regard to these numbers is whether there exist PT-numbers of the form $3^k p$ with p an odd prime and $k \geq 4$ (see [5]).

In the literature, there are many generalizations of the totient function (for example, see [2, 3, 6, 8] and the special chapter on this topic in [7]). Among these, the most significant is probably the *Jordan's totient function* (see [1]). Given a finite group G, in [10, 11, 12, 13] we introduced and studied the function

$$\varphi(G) = |\{a \in G \mid o(a) = \exp(G)\}|.$$

It is obvious that $\varphi(\mathbb{Z}_n) = \varphi(n)$, for all $n \in \mathbb{N}^*$, and so another generalization of the classical function φ is obtained.

Inspired by Definition 1.1 and the above generalization of the totient function, we came up with the following group-theoretical concept.

Definition 2. *A finite group G is called a perfect totient group (or a PT-group, in short) if*

$$|G| = \sum_{i=1}^{c_G} \varphi^i(G),$$

where $c_G = \min\{m \in \mathbb{N}^* \mid \varphi^m(G) = 1\}$.

This concept is natural since the finite cyclic group $(\mathbb{Z}_n, +)$ is a PT-group if and only if the positive integer n is a PT-number. The main goal of the current note is to establish some results about PT-groups.

Most of our notation is standard and will not be repeated here. Some basic definitions and results of group theory can be found in [4, 9].

2. Main results

By Section 4 of [10], we know that there exist large classes of groups G with $\varphi(G) \neq 0$. Clearly, PT-groups also satisfy this property. Another interesting property of PT-groups is the following.

Theorem 3. *All PT-groups are solvable.*

Proof. Let G be a PT-group. Obviously, if G is trivial, then it is solvable. So, we can assume that G is not trivial and let k be the number of cyclic subgroups of order $\exp(G)$ of G. If $\exp(G) \neq 2$, then

$$\varphi(G) = |\{a \in G \mid o(a) = \exp(G)\}| = k\varphi(\exp(G))$$

is even. It follows that $\varphi^i(G)$ is even, for all $1 \leq i \leq c_G - 1$, and so

$$|G| = \sum_{i=1}^{c_G} \varphi^i(G)$$

is odd. Thus G is solvable.

If $\exp(G) = 2$, then G is an elementary abelian 2-group which is solvable, as desired. $\square$

The next theorem shows that there exist non-cyclic PT-groups whose orders are PT-numbers.

Theorem 4. *The following groups are PT-groups:*

a) *The direct product $G = \mathbb{Z}_{3^{n_1}} \times \cdots \times \mathbb{Z}_{3^{n_r}}$, where $r \geq 2$ and $n_{r-1} < n_r$.*

b) *The modular group 3-group $M(3^n) = \langle x, y \mid x^{3^{n-1}} = y^3 = 1,\ y^{-1}xy = x^{3^{n-2}+1}\rangle$, where $n \geq 3$.*

Proof. a) Theorem 2.3 of [10] shows that $\varphi(G) = 2 \cdot 3^{n-1}$, where $n = n_1 + \cdots + n_r$. Then

$$\varphi^i(G) = 2 \cdot 3^{n-i}, \text{ for all } 1 \leq i \leq n,$$

implying that $c_G = n + 1$ and

$$\sum_{i=1}^{c_G} \varphi^i(G) = 1 + \sum_{i=1}^{n} 2 \cdot 3^{n-i} = 3^n = |G|,$$

as desired.

b) The result follows from a) since $M(3^n)$ has the same cyclic subgroup structure as $\mathbb{Z}_3 \times \mathbb{Z}_{3^{n-1}}$ (see e.g. [9], vol. II).

$\square$

We observe that $\mathbb{Z}_3^2$ is not a PT-group because $c_{\mathbb{Z}_3^2} = 4$ and

$$\sum_{i=1}^{4} \varphi^i(\mathbb{Z}_3^2) = 8 + 4 + 2 + 1 = 15 \neq 9.$$

Also, we observe that $G = \mathbb{Z}_3 \times \mathbb{Z}_9$ is a PT-group by Theorem 2.2. Since G contains both a subgroup and a quotient isomorphic with $\mathbb{Z}_3^2$, we infer the following proposition.

Proposition 5. *The class of PT-groups is not closed under subgroups or quotients.*

Let $G_1 = G \times G = \mathbb{Z}_3^2 \times \mathbb{Z}_9^2$. Then $c_{G_1} = 8$ and $\varphi(G_1) = 2^3 \cdot 3^4$ by Theorem 2.3 of [10], implying that

$$\sum_{i=1}^{8} \varphi^i(G_1) = 2^3 \cdot 3^4 + 2^3 \cdot 3^3 + ... + 2^3 \cdot 3^0 + 2^2 + 2^1 + 1 = 2^2 \cdot 3^5 + 3 \neq 3^6.$$

Thus, we have the following result:

Proposition 6. *The class of PT-groups is not closed under direct products or extensions.*

Finally, we propose the following problem.

Open problem. *Are there* (*non-cyclic*) *PT-groups whose orders are not PT-numbers*?

Note that we searched groups of the following types

a) $G = \mathbb{Z}_p \times \mathbb{Z}_{pq}$, where $p \neq q$ are primes and $\varphi(G) = (p^2-1)(q-1)$

b) $G = \mathbb{Z}_p \times (\mathbb{Z}_q \rtimes \mathbb{Z}_p)$, where p, q are primes, $p \mid q-1$ and $\varphi(G) = (p-1)(q-1)$

c) $G = \mathbb{Z}_p \times (\mathbb{Z}_p \rtimes \mathbb{Z}_q)$, where p, q are primes, $q \mid p-1$ and $\varphi(G) = p(p-1)(q-1)$,

but without any result.

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