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# ALGEBRAIC CLASSIFICATION AND RIGIDITY PROPERTIES OF SOLVABLE ABC GROUP ACTIONS ON THE THREE DIMENSIONAL TORUS

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**Abstract.** We study algebraic classification and rigidity properties of ABC group actions on the three torus $\mathbb{T}^3$, by linear and affine transformations. The linear part of such an action is an ABC subgroup of $SL(3, \mathbb{Z})$. We investigate when such a linear ABC action on $\mathbb{T}^3$ can be extended to an affine action that has no identity factors. For a particular class of such actions, we show KAM rigidity; the main reason for the existence of the conjugacy is KAM rigidity of the parabolic $\mathbb{Z}^2$ action inside the ABC group action.

## 1. Introduction

Over the last three decades, many papers exhibited *local rigidity* of different smooth group actions on manifolds. Most known examples include (partially) hyperbolic $\mathbb{Z}^k$ or $\mathbb{R}^k$ actions, for $k > 1$, isometric or actions of lattices of groups with property (T). For solvable non-abelian groups there are very few results. One is the local rigidity of analytic actions by Baumslag-Solitar group $BS(1, k)$, $k \geq 2$, on the circle $\mathbb{T}$ studied in [BW04]. They also classify all the conjugacy classes of such faithful actions. Another is local rigidity of Heisenberg group action on the torus in [HSW18]. See [Fis07] for a background and a very extensive overview of the local rigidity problem for general group actions.

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In this paper, we study Abelian-by-cyclic (ABC for short) group actions on the torus $\mathbb{T}^3$, generated by toral automorphisms and their affine extensions, and we investigate their local rigidity properties. We use KAM (Kolmogorov - Arnold - Moser) method to do so, which was initially developed by Kolmogorov in 1950's, whose ideas were carried out in detail by Arnold and Moser. KAM method was developed initially to answer the question of the stability of our solar system, which was one of the problems Poincaré itself tried to answer in 1890's, but he run into the main obstacle in the study, namely the so called small divisors, which was finally overcome by ideas of Kolmogorov, Arnold and Moser (see [Kol54]). Later KAM theory found applications in many parts of mathematics.

To our knowledge, there are very few local rigidity results for non-abelian group actions on manifolds that relie on the KAM method, and none of them rely on the study of non-abelian group relations and cohomology associated to the whole group action. Instead, they use abelian part of the action and cohomology over that abelian part, as it is done in [WX20] for ABC groups whose abelian part is generated by simultaneously Diophantine translations.

Despite the algebraic classification of ABC group actions on $\mathbb{T}^3$ by linear and affine transformation, our main goal here is to study local rigidity properties of non-abelian group actions where one is forced to genuinely use and analize noncommutative group relations over the whole group action.

As it is well known, a finitely presented, torsion-free ABC group $\Gamma$ admits a short exact sequence

$$0 \hookrightarrow \ Z^d \to \Gamma \to \mathbb{Z} \to 0$$

and is of the form $\Gamma = \Gamma_B \cong \mathbb{Z} \ltimes_B \mathbb{Z}^d$, where $B = (b_{ij})$ is an integer-valued, $d \times d$ invertible matrix, and

$$\Gamma_B = \left\langle g_0, g_1, \ldots, g_d \,\middle|\, g_0 g_i = \left(\prod_{j=1}^{d} g_j^{b_{ji}}\right) g_0, g_i g_j = g_j g_i, i, i = 1, \ldots, d \right\rangle.$$

ABC groups have been a very active subject of study in geometric group theory. The classification of ABC groups, up to quasi-isometry, has been done in [FM00], for non-polycyclic case: $|\det B| > 1$, and in [EFW12], [EFW13], for polycyclic case: $\det B = \pm 1$. In [FM99] and [EF10] are also given some connections between geometry and dynamics of these groups.

Here we deal with actions of ABC polycyclic groups, and for simplicity, we assume that $\det B = 1$. We also restrict ourselves to the case $d = 2$, i.e. $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a regular $2 \times 2$ integer-valued matrix with $ad - bc = 1$. Hence, from now on

$$\Gamma = \Gamma_B = \left\langle g_0, g_1, g_2 \,\middle|\, g_0 g_1 = g_1^a g_2^c g_0, g_0 g_2 = g_1^b g_2^d g_0, g_1 g_2 = g_2 g_1 \right\rangle.$$

A $C^r$ action $\rho$ of $\Gamma$ on $\mathbb{T}^3$ is a homomorphism of groups

$$\rho \colon \Gamma \to \mathrm{Diff}^r(\mathbb{T}^3),$$

and it is completely determined by its values on generators $\rho(g_0)$, $\rho(g_1)$ and $\rho(g_2)$. Here $\mathrm{Diff}^r(\mathbb{T}^3)$ denotes the group of orientation-preserving, $C^r$ diffeomorphisms of $\mathbb{T}^3$. In [WX20] authors show, using Zimmer amenable reduction theorem, and results from [HK87] and [FKS13], under the assumption of $B$ being hyperbolic, given any action $\rho \colon \Gamma_B \to \mathrm{Diff}^r(\mathbb{T}^N)$ all the Lyapunov exponents of $\rho(g_i)$ vanish, with respect to any invariant measure. This subexponential growth implies that the $\mathbb{Z}^2$ part of the action is homotopic to a $\mathbb{Z}^2$ unipotent action.

Let us now introduce some general terminology that will be useful in the coming sections. Let $G$ be a finitely generated group that acts on a smooth compact manifold $M$ and let $r \geq 1$. We introduce a metric on the set $\mathcal{A}^r(G, M)$ of all $C^r$ $G$ actions on M in the following way. If a fixed set $S$ generates $G$, then we define

$$d_r(\rho_1, \rho_2) = \sup_{g \in S} d_{C^r}(\rho_1(g), \rho_2(g)),$$

for any two actions $\rho_1, \rho_2 \in \mathcal{A}^r(G, M)$, where $d_{C^r}$ is the standard metric on the space $\mathrm{Diff}^r(M)$. Of course, this metric is not independent of the generating set $S$, but given any two generating sets of $G$, the associated metrics are equivalent. In this way, we introduced topology on the set of all group actions. This set is, in fact, a topological group, meaning that the composition and inversion in $\mathcal{A}^r(G, M)$ are continuous group operations. To simplify our exposition and proofs, we deal only with infinitely differentiable group actions.

Let $\pi \colon \mathbb{R}^d \to \mathbb{T}^d$ be the standard projection map. We use the $\ell^1$ norm on $\mathbb{Z}^d$, i.e., for a vector $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ we define its

$\ell^1$ norm as $|k| = |k_1| + \ldots + |k_d|$. Now we define $D_j = \frac{\partial}{\partial x_j}$, for $j = 1, \ldots, d$ and $D^k = D_{k_1} \cdots D_{k_d}$, for $k \in \mathbb{N}_0^d$. On $\mathrm{C}^\infty(\mathbb{T}^\mathrm{d})$, and hence on $\mathrm{Diff}_0^\infty(\mathbb{T}^\mathrm{d})$, we introduce a sequence of $\mathrm{C}^r$ norms

$$g_r = \max_{1 \le j \le d} \max_{|k| \le r} \sup_{x \in \mathbb{T}^d} \left| D^k g_j(x) \right|,$$

where $g = (g_1, \ldots, g_d) \in \mathrm{C}^\infty(\mathbb{T}^\mathrm{d})$ and the maximum in the middle is taken over all vectors $k \in \mathbb{N}_0^d$, for which $|k| \le r$.

In our case of smooth $\Gamma$ action on $\mathbb{T}^3$ the metric will be induced by the sequence of norms on $C^r$.

1.1. **Organization of the paper.** First, in the next section we study algebraic classification of $\Gamma$ actions on $\mathbb{T}^3$ by linear transformations and their affine extensions. We obtain conditions under which these extensions exist on $\mathbb{T}^3$. Next, in section 3 we study with rigidity properties of such admissible actions on $\mathbb{T}^3$.

## 2. Algebraic classification of $\Gamma$ actions

2.1. **Linear $\Gamma$ actions.** Here we study $\Gamma$ actions on $\mathbb{T}^3$ by toral automorphisms which represent all the possible homotopy classes of smooth $\Gamma$ actions $\rho\colon \Gamma \to \mathrm{Diff}^r(\mathbb{T}^3)$. Let $\rho_L$ denote a linear $\Gamma$ action on the torus $\mathbb{T}^3$ by torus automorphisms, i.e. let $\rho_L\colon \Gamma \to \mathcal{A}ut(\mathbb{T}^3)$ be a homomorphism, where $\mathcal{A}ut(\mathbb{T}^3)$ denotes the group of automorphisms of $\mathbb{T}^3$. Hence, the elements $\rho_L(g_0)$, $\rho_L(g_1)$ and $\rho_L(g_2)$ act on $\mathbb{T}^3$ by matrices from $SL(3, \mathbb{Z})$, which we denote by $F$, $U$ and $V$ respectively, i.e.

$$\begin{aligned} \rho_L(g_0)x &= Fx \\ \rho_L(g_1)x &= Ux \\ \rho_L(g_2)x &= Vx, \end{aligned} \tag{1}$$

for any $x \in \mathbb{T}^3$. Because of the subexponential growth described earlier, we let $U$ and $V$ be parabolic matrices, and moreover, up to a change of coordinates, we may take that they are upper triangular. Hence, up to a change of coordinates

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $V = X$ or $V = Y$, where

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

However, note that we will not assume that $B$ is hyperbolic. We only assume $B \in SL(2, \mathbb{Z})$. We have the following relations between matrices $F$, $U$, and $V$

$$\begin{aligned} FU &= U^a V^c F \\ FV &= U^b V^d F \\ UV &= VU. \end{aligned} \tag{2}$$

These relations from $\Gamma$ impose some restrictions to what $F$ can be. We describe these restrictions in the following Proposition.

**Proposition 1.** *Let $\rho_L$ be the $\Gamma$ action on $\mathbb{T}^3$ by toral automorphisms, given by (1). Up to a change of coordinates, we have one of the following two cases:*

*1. $F = Z := \begin{bmatrix} B & O \\ O & 1 \end{bmatrix}$ where $V = X$,*

*2. $F = W := \begin{bmatrix} 1 & O \\ O & \tilde{B} \end{bmatrix}$ where $V = Y$ and $\tilde{B} = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$.*

*Proof.* Let $F = [f_{ij}]_{i,j=\overline{1,3}} \in SL(3, \mathbb{Z})$. In the first case where $V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, the first group relation from (2) $FU = U^a V^c F$ forces $f_{31} = f_{32} = 0$. Since $F \in SL(3, \mathbb{Z})$, without loss of generality we can assume that $f_{33} = 1$. After plugging in these values for coefficients of $F$, we obtain $f_{11} = a$ and $f_{21} = c$. From the second relation in (2) $FV = U^b V^d F$ we get that $f_{12} = b$ and $f_{22} = d$, which proves 1. from the proposition.

In the case where $V = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, the first group relation from (2) $FU = U^a V^c F$ now forces $f_{12} = f_{13} = 0$, and hence, without loss of generality we can assume that $f_{11} = 1$. Using the same relation once again, we conclude that $f_{22} = a$ and $f_{32} = -c$. From the second relation in (2) $FV = U^b V^d F$ we get that $f_{23} = -b$ and $f_{33} = d$, which proves 2. from the proposition. ∎

**Corollary 2.** *Any smooth* $\Gamma$ *action* $\rho\colon \Gamma \to \mathrm{Diff}^r(\mathbb{T}^3)$ *on* $\mathbb{T}^3$ *is, up to a change of coordinates, homotopic to one of the two linear models described in the previous proposition.*

2.2. **Affine** $\Gamma$ **actions.** We also study the algebraic classification of *affine extensions* of $\Gamma$ actions

$$\rho_\tau\colon \Gamma \to \mathrm{Aff}(\mathbb{T}^3).$$

Here $\mathrm{Aff}(\mathbb{T}^d)$ denotes the group of all affine transformations of $\mathbb{T}^d$. These actions are generated by automorphisms of $\mathbb{T}^3$, followed by toral translations $T_\omega(x) := x + \omega$, for every $x \in \mathbb{T}^3$ and some fixed $\omega \in \mathbb{T}^3$. To be more concrete, let $\alpha, \beta$ and $\gamma$ be three vectors from $\mathbb{T}^3$. The action is completely determined by its values on generators $F_\alpha := \rho_\tau(g_0), U_\beta := \rho_\tau(g_1), V_\gamma := \rho_\tau(g_2)$, where

$$\begin{aligned} F_\alpha(x) &= Fx + \alpha \\ U_\beta(x) &= Ux + \beta \\ V_\gamma(x) &= Vx + \gamma, \end{aligned} \tag{3}$$

for any $x \in \mathbb{T}^3$. Note that the linear part of these actions is given by the action $\rho_L$, defined in (1).

**Definition 3.** *An affine action of* $\Gamma$ *on* $\mathbb{T}^d$ *is a homomorphism* $\rho\colon \Gamma \to \mathrm{Diff}^\infty(\mathbb{T}^d)$*, which can be represented as* $\rho(g) = \rho_L(g) + T_{\alpha(g)}$*,* $g \in \Gamma$*, where the homomorphism* $\rho_L\colon \Gamma \to SL(d,\mathbb{Z})$ *is called the linear part of the action, and for* $g \in \Gamma$*,* $T_{\alpha(g)}\colon \mathbb{T}^d \to \mathbb{T}^d$ *is the translation by* $\alpha \in \mathbb{T}^d$*. The action* $g \mapsto T_{\alpha(g)}$ *is called the translation part of the action* $\rho$*.*

In order for such a $\Gamma$ action to exist, the group relations (2) impose restrictions on vectors $\alpha, \beta$, and $\gamma$ from (3). We describe those restrictions in the following propositions, omitting the proof since it is a straightforward calculation.

**Proposition 4.** *Let* $\rho_\tau$ *be an affine* $\Gamma$ *action on* $\mathbb{T}^3$ *generated by (3), where* $\alpha, \beta, \gamma \in \mathbb{T}^3$ *are fixed vectors. Then the following equalities must hold:*

$$\begin{aligned} F\beta + \alpha &= U^a V^c \alpha + \sum_{j=0}^{c-1} U^a V^j \gamma + \sum_{j=0}^{a-1} U^j \beta \\ F\gamma + \alpha &= U^b V^d \alpha + \sum_{j=0}^{d-1} U^b V^j \gamma + \sum_{j=0}^{b-1} U^j \beta \\ (U - Id)\gamma &= (V - Id)\beta. \end{aligned} \tag{4}$$

Note that the restrictions on $\alpha, \beta$ and $\gamma$ in the previous proposition do not depend on the specific forms of $F$, $U$, and $V$. They only depend on group relations from $\Gamma$, hence on matrix $B$. If we now restrict ourselves according to the two cases in Proposition 1, we get more specific restrictions on $\alpha, \beta$ and $\gamma$. We describe these restrictions in the next proposition.

**Proposition 5.** *Let $\rho_\tau$ be the affine $\Gamma$ action on $\mathbb{T}^3$ generated by (3), where $\alpha, \beta, \gamma \in \mathbb{T}^3$ are fixed, and where $F$, $U$ and $V$ are given in Proposition 1. Then, if we write $\bar{v} = (v_1, v_2)$ for a general vector $v = (v_1, v_2, v_3) \in \mathbb{T}^3$, up to a change of coordinates, we have one of the two following cases:*

$$\begin{aligned} & F = Z, V = X \\ & B\bar{\beta} = \bar{\alpha} + \alpha_3 \begin{bmatrix} a \\ c \end{bmatrix} + c\bar{\gamma} + a\bar{\beta} \\ & B\bar{\gamma} = \bar{\alpha} + \alpha_3 \begin{bmatrix} b \\ d \end{bmatrix} + d\bar{\gamma} + b\bar{\beta} \\ & \gamma_3 = \beta_3 = 0, \end{aligned} \tag{5}$$

*or*

$$\begin{aligned} & F = W, V = Y \\ & F\beta + \alpha = U^a V^c \alpha + \sum_{j=0}^{c-1} U^a V^j \gamma + \sum_{j=0}^{a-1} U^j \beta \\ & F\gamma + \alpha = U^b V^d \alpha + \sum_{j=0}^{d-1} U^b V^j \gamma + \sum_{j=0}^{b-1} U^j \beta \\ & \gamma_3 = \beta_2. \end{aligned} \tag{6}$$

We denote by $\mathcal{A}_1$ the set of all triples of vectors $\tau = (\alpha, \beta, \gamma)$ that satisfy (5) and by $\mathcal{A}_2$ the set of all triples of vectors $\tau = (\alpha, \beta, \gamma)$ that satisfy (6).

In the first case of Proposition 5, there is a substantial difference between local rigidity properties of the following two subcases: $B$ is *not traceless* and $B$ is *traceless*. The first case forces also $\alpha_3 = 0$ in $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, while the second case gives $\alpha_3 \neq 0$. This means that in the first subcase, any affine $\Gamma$ action has an identity factor. We denote by $\mathcal{A}_{tr(B)\neq 0}$ and $\mathcal{A}_{tr(B)=0}$ the two disjoint subsets of $\mathcal{A}_1$, corresponding to the two above subcases, respectively. Similarly, in the second case of Proposition 3, the two different subcases are: $a - b \neq 1$ and $a - b = 1$. We denote by $\mathcal{A}_{a-b\neq 1}$ and $\mathcal{A}_{a-b=1}$ the corresponding disjoint subsets of $\mathcal{A}_2$, respectively. In these two subcases, not any affine $\Gamma$ action has an identity factor, but the corresponding $\mathbb{Z}^2$ subaction generated by $U_\beta$ and $V_\gamma$ always has an identity factor. The previous discussion motivates the next definition, which is taken from [DFS23].

**Definition 6.** *Let $G$ be a group and let $\rho_L\colon G \to \mathcal{A}ut(\mathbb{T}^d)$ be an action on $\mathbb{T}^d$ by toral automorphisms. $\rho_L$ is said to be locked if any affine $G$ action $\rho_\tau\colon G \to \mathrm{Aff}(\mathbb{T}^d)$ on $\mathbb{T}^d$, with linear part $\rho_L$, has an identity factor.*

Combining all the previous results and discussion, we obtain the following classification according to the previous definition.

**Proposition 7.** *Let $\rho_L$ be the $\Gamma$ action on $\mathbb{T}^3$ by toral automorphisms, given by (1). Up to a change of coordinates, we have one of the following three cases:*

1. *Case I: the linear part $\rho_L$ of an affine action is locked, and its $\mathbb{Z}^2$ linear subaction is also locked inside $\Gamma$,*
2. *Case II: the linear part $\rho_L$ of an affine action is not locked, but its $\mathbb{Z}^2$ linear subaction is locked inside $\Gamma$,*
3. *Case III: the linear part $\rho_L$ of an affine action is not locked, and its $\mathbb{Z}^2$ linear subaction is also not locked inside $\Gamma$.*

As a consequence of the previous proposition, we conclude that any affine $\Gamma$ action $\rho_\tau$ in Case I is not minimal and not ergodic, with respect to the volume. However, the situation in the other two cases is completely different. We say that a vector $\tau = (\tau_1, \ldots, \tau_d) \in \mathbb{R}^d$ is irrational if the numbers $\tau_1, \ldots, \tau_d$ are rationally independent. And we say that the affine $\Gamma$ action $\rho_\tau$ on $\mathbb{T}^d$ is irrational if the vector $\tau$ is irrational.

**Proposition 8.** *Any irrational affine $\Gamma$ action $\rho_\tau$ on $\mathbb{T}^3$ from Case II and Case III is minimal and ergodic, with respect to the volume.*

A proof of the previous proposition follows from the irrationality of the translation parts of the action and their topological and ergodic properties.

2.3. **On rigidity properties of $\Gamma$ actions on $\mathbb{T}^3$.** In this section, we study local rigidity properties of smooth $\Gamma$ actions on $\mathbb{T}^3$, generated by toral automorphisms and affine maps in Case III of Proposition 7. Rigidity properties of Case I and Case II are studied in [P25], where a form of local rigidity is obtained in those cases using KAM method and analysis of cohomology over the acting group and truly using the noncommutative relations of the acting group. To our knowledge, that was the first time the analysis of noncommutative relations was exploited together with KAM machinery.

**Definition 9.** *A smooth $G$ action $\rho\colon G \to \mathrm{Diff}^\infty(M)$ is $C^\infty$ locally rigid if for any smooth action $\tilde{\rho}\colon G \to \mathrm{Diff}^\infty(M)$, which is close enough to $\rho$ in some $C^r$ topology, there exists a diffeomorphism $h \in \mathrm{Diff}^\infty(M)$ that conjugates $\tilde{\rho}$ to $\rho$, i.e.*

$$\tilde{\rho}(g, h(x)) = h(\rho(g, x)),$$

*for any $g \in G$ and any $x \in M$.*

This is a very strong property which holds for some Anosov $\mathbb{Z}^k$ actions [KS97]. In the Case I of Proposition 7 of affine parabolic and partially hyperbolic actions of ABC groups, this does not hold in this full generality, but as we already mentioned, some local classification is still possible. Without any further assumptions made on the group action, it is not common that the action is locally rigid. However, they can exhibit some weaker forms of local rigidity, provided some constraints are met. These constraints are usually stated by fixing some dynamical quantities that characterize the unperturbed action, e.g. existence of an invariant measure, zero averages of (some and hence all) acting group elements with respect to some invariant measure, etc. We define now the notion of *KAM rigidity* which is central to this paper. The definition of KAM rigid affine actions is given in [DFS23].

**Definition 10.** *A smooth affine $\Gamma$ action $\rho$ on $\mathbb{T}^d$ is KAM rigid if any smooth volume-preserving perturbation $\tilde{\rho}$, which is close to $\rho$ in some $C^r$ topology and with the average zero of $\tilde{\rho} - \rho$ on group generators, with respect to the invariant volume, is $C^\infty$ conjugate to $\rho$.*

A slightly less general form of rigidity is proved in [P25] in the Case I of Proposition 7, the so called fiberwise KAM rigidity, where the perturbations of such actions are only allowed in fibers, and not in the base (if we see the torus $\mathbb{T}^3$ as $\mathbb{T}^2$ fibered over $\mathbb{T}$) since there is identity factor of the action. This was done using KAM tehniques and intensive study of the cohomological equations corresponding to noncommutative relations coming from the group action.

The study of local rigidity of smooth abelian group actions by KAM techniques started with Moser in the '90s, in the context of smooth $\mathbb{Z}^n$, group actions, $n > 1$, of orientation-preserving diffeomorphisms of the circle $\mathbb{T}$, under an appropriate arithmetic condition on rotation numbers of the action generators [Mos90].

**Definition 11.** *If vectors $\alpha_1, \dots, \alpha_m \in \mathbb{T}^d$ satisfy*

$$\max_{j=1\dots,m} |\langle \alpha_j, k\rangle - l| > \frac{C}{|k|^\delta}, \tag{7}$$

*for all $(k, l) \in \mathbb{Z}^d \times \mathbb{Z} \setminus \{\mathbf{0}\}$ for some constants $C > 0$ and $\delta > 0$, then we say that they are simultaneously $(C, \delta)$-Diophantine.*

It is a standard fact that the set of simultaneously Diophantine vectors has full Lebesgue measure on $\mathbb{T}^d$.

Moser showed in [Mos90] local rigidity of $\mathbb{Z}^n$, $n \geq 1$, generated by circle translations $T_{\omega_1}, \dots, T_{\omega_n}$, provided the rotation numbers $\omega_1, \dots, \omega_n$ are simultaneously Diophantine. This result was later extended to the higher dimensional cases, where instead of rotation number, the notion of *rotation vector* of toral homeomorphisms is may be defined. One speaks of a rotation set instead. And the situation is the same, even if this set is a singleton since this object does not represent a complete topological invariant, contrary to the rotation number in the one-dimensional case. See [FRH05], [DF19], [Kar18], [WX20] and [P21].

Small smooth perturbations $\rho$ of the affine $\Gamma$ actions $\rho_\tau$ on $\mathbb{T}^3$ perturbations are generated by maps of the form

$$\tilde{F}_\alpha = F_\alpha + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}, \tilde{U}_\beta = U_\beta + \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}, \tilde{V}_\gamma = V_\gamma + \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}, \tag{8}$$

where $f_i, g_i, k_i \colon \mathbb{T}^3 \to \mathbb{T}$, $i = 1, 2, 3$, are some small functions in some $C^r$ norm.

Using Definition 10, a corollary of Moser's local rigidity result can be stated in the following way.

**Theorem 12** (Moser 1990). *Let $n \geq 1$ and $d \geq 1$ be integers. Any smooth Diophantine affine $\mathbb{Z}^n$ action on $\mathbb{T}^d$, homotopic to the identity, is KAM rigid.*

2.3.1. **KAM rigidity: Case III of Proposition 7.** In Case III, the situation is very different from Case I and Case II of Proposition 7, in the sense that neither the linear part $\rho_L$ of the affine $\Gamma$ action nor the linear part of the $\mathbb{Z}^2$ subaction is locked inside $\Gamma$. For that reason, one can first show that there is a full measure set of affine $\mathbb{Z}^2$ actions on $\mathbb{T}^3$, generated by $U_\beta$ and $V_\gamma$, that are KAM rigid. This is done in [DFS23]. We state a corollary that follows from that result applied to our affine $\Gamma$ action on $\mathbb{T}^3$.

**Corollary 13.** [DFS23] *There is a full measure set of affine $\Gamma$ actions $\rho_\tau$ on $\mathbb{T}^3$, from Case III, for which the $\mathbb{Z}^2$ affine subaction of $\rho_\tau$ is KAM rigid.*

Using ergodicity, but only of the $\mathbb{Z}^2$ subaction, and noncommuting relations in the group $\Gamma$, we get that the same conjugacy, as for the $\mathbb{Z}^2$ subaction in the previous theorem, also conjugates the whole affine $\Gamma$ action $\rho_\tau$ to some other affine $\Gamma$ action $\rho_{\tau'}$ from Case III, where $\tau' = (\alpha', \beta, \gamma)$ and the vector $\alpha'$ will be computed explicitly. Thus, the KAM method used here relies completely on the abelian part of the action. We prove the following theorem.

**Theorem 14.** *There is a full measure set of affine $\Gamma$ actions $\rho_\tau$ on $\mathbb{T}^3$, from Case III of Proposition 7, for which any sufficiently $C^r$ small perturbed action, for some $r \geq 1$, is smoothly conjugate to an affine action $\rho_{\tau'}$ from Case III, where $\tau' = (\alpha', \beta, \gamma)$ and $\alpha'$ is of the form $(p_1, \alpha_2, \alpha_3) \in \mathbb{T}^3$.*

*Proof.* Let $H$ be the smooth conjugacy obtained from Corollary 13 which conjugates the perturbed $\mathbb{Z}^2$ subaction, generated only by $\tilde{U}_\beta$ and $\tilde{V}_\gamma$ from (8), to the unperturbed $\mathbb{Z}^2$ subaction generated by $U_\beta$ and $V_\gamma$ from (3). Using group relations, we show that the conjugacy $H$ also conjugates $\tilde{F}_\alpha$ from (8) to a smooth map of the form $F_{\alpha'}$. Define $G = H^{-1} \circ \tilde{F}_\alpha \circ H$ and write $G$ as $G = F + p$ for some smooth function $p$. Since $\tilde{F}_\alpha, \tilde{U}_\beta, \tilde{V}_\gamma$ is conjugate to $G, U_\beta, V_\gamma$ via $H$, we have that $G, U_\beta, V_\gamma$ also satisfy the group relations (2). In other words

$$GU_\beta = U_\beta^a V_\gamma^c G$$
$$GV_\gamma = U_\beta^b V_\gamma^d G,$$

which is equivalent to

$$p \circ U_\beta + F\beta = U^a V^c p + \sum_{j=0}^{c-1} U^a V^j \gamma + \sum_{j=0}^{a-1} U^j \beta$$
$$p \circ V_\gamma + F\gamma = U^b V^d p + \sum_{j=0}^{d-1} U^a V^j \gamma + \sum_{j=0}^{b-1} U^j \beta.$$

Since $(\alpha, \beta, \gamma) \in \mathcal{A}_{a-b=1}$, (6) from Proposition 5 implies

$$p \circ U_\beta = U^a V^c p - (Id - U^a V^c)\alpha = -\alpha + U^a V^c (p + \alpha)$$
$$p \circ V_\gamma = U^b V^d p - (Id - U^b V^d)\alpha = -\alpha + U^b V^d (p + \alpha).$$

Since $\mathbb{Z}^2$ subaction is ergodic with respect to the volume this forces $p_2$ and $p_3$ to be constant functions. Averaging on both sides of previous equations and using that the $\mathbb{Z}^2$ subaction preserves the volume we get

$$Ave(p_1) = Ave(p_1) + cAve(p_2) + aAve(p_3) - c\alpha_2 - a\alpha_3$$
$$Ave(p_1) = Ave(p_1) + dAve(p_2) + bAve(p_3) - d\alpha_2 - b\alpha_3.$$

Therefore, $p_2 = \alpha_2$, $p_3 = \alpha_3$ and hence

$$p_1 \circ U_\beta = p_1.$$

Using ergodicity once again, we conlcude that $p_1$ is also a constant. Therefore, $G$ is is of the form $F_{\alpha'}$ where $\alpha' = (p_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$. This finishes the proof. □

The full measure set in the previous theorem comes from the Diophantine condition (7), but this time only on $\beta$ and $\gamma$, and not on the whole vector $\tau$. Note that in this case, if the perturbed $\mathbb{Z}^2$ subaction, generated by $\tilde{U_\beta}$ and $\tilde{V_\gamma}$, is measure preserving, then so is $\tilde{F_\alpha}$, since the $\mathbb{Z}^2$ subaction is uniquely ergodic. This follows from group relations in $\Gamma$. If in addition $\int f = 0$ then $\tilde{F_\alpha}$ is $C^\infty$-conjugated to the original $F_\alpha$ via the same conjugacy $H$ as in the previous theorem. Hence, we obtain the following corollary.

**Corollary 15.** *There is a full measure set of affine $\Gamma$ actions $\rho_\tau$ on $\mathbb{T}^3$, from Case III, that are KAM rigid.*

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