"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 35 (2025), No. 1, 47 - 68

ALMOST ALTERING DISTANCE FUNCTIONS AND COMMON FIXED POINT THEOREMS FOR TWO PAIRS OF MAPPINGS SATISFYING A NEW TYPE OF COMMON LIMIT RANGE PROPERTY IN SEMIMETRIC SPACES

VALERIU POPA AND MARCELINA MOCANU

Abstract. In this paper, we prove a general common fixed point theorem for two pairs of mappings satisfying a new type of common limit range property in semimetric (symmetric) spaces. Our result relies on the use of an almost altering distance and of implicit relations and generalizes known results. As applications, we obtain new fixed point theorems for mappings satisfying contractive conditions of integral type and for c-class functions.

1. Introduction and preliminaries

Classical fixed point results are usually formulated in metric spaces and rely essentially on the triangle inequality. However, in many applications this condition is too restrictive, while the first two axioms of a distance are still preserved. This motivates the study of fixed point theory in semimetric (symmetric) spaces, which generalize metric spaces but retain a meaningful notion of convergence. To overcome the lack of the triangle inequality, additional regularity assumptions such as 1-continuity and condition (H.E.) are commonly employed.

Keywords and phrases: common fixed point, semimetric space, 1-continuous semimetric, almost altering distance function, common limit range property
(2020) Mathematics Subject Classification: 54H25, 47H10.

On the other hand, various generalizations of contractive conditions have been developed to unify and extend classical results. Altering distance functions and their recent extension to almost altering distances provide a flexible framework that includes contractive conditions of integral type and nonlinear inequalities. Furthermore, the common limit range property has proved to be an effective substitute for continuity and strong compatibility assumptions in the study of common fixed points.

Motivated by these developments, this paper introduces a new variant of the common limit range property and combines it with implicit contractive conditions that involve almost altering distance functions in the setting of semimetric spaces. The results obtained generalize several known common fixed point theorems and yield new applications to mappings satisfying integral-type contractions and conditions involving C-class functions.

Definition 1.1. *For a non-empty set X, a function $d : X \times X \to \mathbb{R}_+$ is called semimetric on X if the following two conditions hold:*
(S1) $d(x, y) = 0$ if and only if $x = y$;
(S2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

The couple (X, d), where d is a semimetric on X will be called a semimetric space. In the literature, semimetric spaces are also called *symmetric spaces*. In this paper, we will use the name "semimetric space", as symmetric space is also the name of an important concept in Riemannian geometry.

As in metric spaces, given a semimetric d on X, open balls are defined and an induced topology τ_d is obtained. This topology is first-countable, but it is not always Hausdorff. Note that there exist semimetrics for which every open ball is not an open set and every closed ball is not a closed set [11, Theorem 5.6]. As d generally does not satisfy the triangle inequality, it is possible to encounter the case where the function $y \mapsto d(x, y)$ is no longer $1-$Lipschitz on X for any $x \in X$.

Example 1.2. *Consider a nonempty finite set $X = \{x_1, x_2, ..., x_m\}$ and a symmetric matrix $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{R})$ with nonnegative entries, having zero entries only on the main diagonal. Define $d(x_i, x_j) = a_{ij}$ for all $i, j \in \{1, 2, ..., m\}$. Then d is semimetric on X. If there exist $i, j, k \in \{1, 2, ..., m\}$ such that $|a_{ij} - a_{ik}| > a_{jk}$, then the function $y \mapsto d(x_i, y)$ is not $1-$Lipschitz on X.*

The definition of a convergent sequence is the same as in metric spaces: *a sequence* $(x_n)_{n\geq 1}$ *in* X *is said to converge to a point* $x \in X$ *if* $\lim_{n\to\infty} d(x_n, x) = 0$.

In order to compensate the lack of the triangle inequality, in the study of convergent sequences, some additional regularity axioms have been introduced for the semimetric.

As mentioned in [11], in 1931 Wilson [32] introduced the following substitutes for the triangle inequality regarding convergent sequences in a semimetric space (X, d): for all $x, y \in X$ and all sequences $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$ and $(z_n)_{n\geq 1}$ in X,

(W1) (The uniqueness of the limit) If $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$, then $x = y$;

(W2) If $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$, then $\lim_{n\to\infty} d(y_n, x) = 0$;

(W3) If $\lim_{n\to\infty} d(x_n, y_n) = 0$ and $\lim_{n\to\infty} d(y_n, z_n) = 0$, then $\lim_{n\to\infty} d(x_n, z_n) = 0$.

In the following, we recall another substitute for the triangle inequality with regard to convergent sequences in a semimetric space, namely the notion of $1-$continuous semimetric, which has been introduced in [12], and the condition (H.E.) satisfied by a semimetric, introduced in [3].

Definition 1.3. *(1) A semimetric* d on X *is said to be* $1-$*continuous if* $\lim_{n\to\infty} d(x_n, x) = 0$ *implies* $\lim_{n\to\infty} d(x_n, y) = d(x, y)$, *for all* $x, y \in X$ *and all sequences* $(x_n)_{n\geq 1}$ *in* X;

(2) A semimetric d on X *is said to satisfy the condition (H.E) if* $\lim_{n\to\infty} d(x_n, x) = 0$ *and* $\lim_{n\to\infty} d(y_n, x) = 0$ *imply* $\lim_{n\to\infty} d(x_n, y_n) = 0$, *for all* $x \in X$ *and all sequences* $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$ *in* X.

More conditions used as a substitute for the triangle inequality and involving convergent sequences are given in [20].

The notion of weakly compatible pair of mappings was introduced by Jungck in 1996 [19] and was used to prove the existence of common fixed points for non-continuous self-mappings of a metric space. The sets of coincidence points of $f, g : X \to X$ is $C(f, g) = \{x \in X : f(x) = g(x)\}$ and the values $w = f(x) = g(x)$ are called coincidence values of f and g. Two weak compatible mappings commute at their coincidence points, as the following definition shows.

Definition 1.4. [19] *Two mappings $f, g : X \to X$ are called weakly compatible if $f(g(x)) = g(f(x))$ for all $x \in C(f, g)$, i.e. for all $x \in X$ such that $f(x) = g(x)$.*

In 2011, Sintunavarat and Kumam [31] introduced the notion of common limit range property for a pair of self-mappings of a fuzzy metric space. In 2012, Imdad, Pant and Chauhan [13] introduced the notion of common limit range property for two pairs of self-mappings of a Menger space and used this property to prove fixed point results. In 2013, Imdad, Chauhan and Kadelburg [15] proved fixed point theorems for weakly compatible pairs of self-mappings of a metric space, possessing a common limit range property and satisfying some generalized $(\psi, \varphi)-$weak contractive condition. The notion of common limit range property has been used in the study of common fixed points for self-mappings of metric spaces [14] and self-mappings of semimetric spaces [10], [17], [23].

Definition 1.5. [13], [15] *Let (X, d) be a semimetric space and A, B, S, T be self-mappings of X. The pairs (A, S) and (B, T) are said to satisfy the common limit range property with respect to (S, T), denoted by (CLR_{ST}), if there exist two sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ in X such that*

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} B(y_n) = \lim_{n\to\infty} T(y_n) =: z,$$

where $z \in S(X) \cap T(X)$.

We introduce a variant of the common limit range property as follows.

Definition 1.6. *Let (X, d) be a semimetric space and A, S, T be self-mappings of X. The pair (A, S) is said to satisfy the common limit range property with respect to T if there exists a sequence $(x_n)_{n\geq 1}$ in X such that*

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} S(x_n) =: z,$$

where $z \in S(X) \cap T(X)$.

Obviously, if two pairs (A, S) and (B, T) satisfy (CLR_{ST}), then (A, S) satisfies the common limit range property with respect to T. The converse is not true for arbitrary B, as the following example shows.

Example 1.7. *Let $X = \mathbb{R}_+$ and $d(x, y) = (x - y)^2$ for $x, y \in X$. Then (X, d) is a semimetric space. But (X, d) is not a metric space,* *as* $d(x,y) - [d(x,z) + d(z,y)] = -2(z-x)(z-y)$ *for all* $x, y, z \in X$, *hence* $d(x,y) > d(x,z) + d(z,y)$ *whenever* $x < z < y$ *or* $y < z < x$.

Define the following self-mappings of X: $A(x) = \frac{1}{2}(x^2+1)$, $S(x) = \frac{1}{2}(x+1)$ *and* $T(x) = x + \frac{1}{4}$. *Then* $S(X) \cap T(X) = \left[\frac{1}{2}, \infty\right)$. *For a sequence* $(u_n)_{n\geq 1}$ *in* X *such that* $\lim\limits_{n\to\infty} u_n = 0$ *we see that* $\lim\limits_{n\to\infty} A(u_n) = \lim\limits_{n\to\infty} S(u_n) = \frac{1}{2} \in S(X) \cap T(X)$, *therefore, the pair* (A,S) *satisfies the common limit range property with respect to* T. *Assume that for some mapping* $B : X \to X$ *the pairs* (A,S) *and* (B,T) *satisfy* (CLR_{ST}). *Then there exist some sequences* $(x_n)_{n\geq 1}$ *and* $(y_n)_{n\geq 1}$ *in* X *such that* $\lim\limits_{n\to\infty} A(x_n) = \lim\limits_{n\to\infty} S(x_n) = \lim\limits_{n\to\infty} B(y_n) = \lim\limits_{n\to\infty} T(y_n) =: t$ *and* $t \in S(X) \cap T(X) = \left[\frac{1}{2}, \infty\right)$. *Then* $\lim\limits_{n\to\infty} (x_n)^2 = \lim\limits_{n\to\infty} x_n = 2t - 1$, *and hence* $t = \frac{1}{2}$. *As* $\lim\limits_{n\to\infty} T(y_n) = t$, *it follows that* $\lim\limits_{n\to\infty} y_n = 4t - 1 = 1$, *while* $\lim\limits_{n\to\infty} B(y_n) = t = \frac{1}{2}$. *If* B *is continuous and* $B(1) \neq \frac{1}{2}$, *we obtain a contradiction, therefore, the pairs* (A,S) *and* (B,T) *do not satisfy* (CLR_{ST}).

Branciari [9] proved a general fixed point theorem for mappings satisfying a contractive condition of integral type. The notion of altering distance function φ has been used by Khan, Swaleh and Sessa [21] in proving fixed point theorems for self-mappings of a complete metric space (X,d), satisfying contractive conditions where a distance $d(x,y)$ is replaced by $\varphi(d(x,y))$. Popa and Mocanu [28] proved a common fixed point theorem for two pairs of occasionally weakly compatible mappings satisfying an implicit relation, using the observation that some contractive conditions of integral type can be reduced to contractive conditions involving an altering distance function. Popa and Patriciu [29] introduced the notion of almost altering distance function in order to formulate contractive conditions used in fixed point theorems for a pair of self-mappings with common limit range property in G-metric spaces.

Definition 1.8. [29] *An almost altering distance function is a function* $\psi : [0,\infty) \to [0,\infty)$ *which satisfies the following conditions:*

(A1) ψ *is continuous on* $[0,\infty)$;

(A2) $\psi(t) = 0$ *if and only if* $t = 0$.

An altering distance function is an increasing almost altering distance function.

Here, the term "increasing" is used in a non-strict sense, as a synonim for "non-decreasing", i.e. $\psi : [0,\infty) \to [0,\infty)$ is called increasing

if $t_1 < t_2$ implies $\psi(t_1) \leq \psi(t_2)$, for all $t_1, t_2 \in [0, \infty)$.
Obviously, there exist almost altering distance functions that are not increasing, e. g. $\psi : [0, \infty) \to [0, \infty)$ defined by $\psi(t) = t$ for $t \in [0, 1]$ and $\psi(t) = \frac{1}{t}$ for $t \in (1, \infty)$.

In 2014, Ansari [4] introduced the notion of $C-$class function, as a control function in contractive conditions.

Definition 1.9. [4] *A mapping $G : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called a $C-$class function if it is continuous and satisfies the following conditions:*
(G1) $G(s, t) \leq s$ for all $s, t \in [0, \infty)$;
(G2) $G(s, t) = s$ implies $s = 0$ or $t = 0$, for all $s, t \in [0, \infty)$.

Some fixed point theorems via $C-$class functions have been proved in [4] in complete metric spaces, in [5] and [8] in the setting of metric spaces and by Ansari and Kumari [6] in semimetric spaces. Ansari and Razani [7] proved fixed point theorems for self-mappings of a complete $b-$metric space, satisfying some general contractive conditions involving an altering distance function or a so-called $C-$class function.

The purpose of this paper is to prove a general common fixed point theorem in the setting of semimetric spaces, for two pairs of mappings satisfying a new type of common limit range property, and a contractive condition involving an almost altering distance and implicit relations. Our main result generalizes Theorem 3.2 in [16]. As applications, some common fixed point results for mappings satisfying contractive conditions of integral type are obtained.

2. Implicit relations

Several fixed point theorems have been unified by using contractive conditions involving implicit functions [24], [25]. The use of implicit contractive conditions in fixed point theorems has the advantage of unifying and generalizing a wide class of classical contractive conditions, usually with simpler proofs.

Imdad and Chauhan [14] used implicit contractive conditions to obtain common fixed point theorems for pairs of self-mappings with the common limit range property, in metric spaces. Similar results, using implicit contractive conditions and the common limit range property, have been proved in the setting of semimetric (symmetric) spaces by Imdad, Sharma and Chauhan [16] and in the setting of metric spaces by Popa [26].

Ali and Imdad [2] introduced the following class of implicit relations.

Definition 2.1. *Let $\mathcal{F}$ be the family of all real lower semicontinuous functions $F : \mathbb{R}_+^6 \longrightarrow \mathbb{R}$ satisfying the following inequalities:*

(F1) $F(t, 0, t, 0, 0, t) > 0$ for all $t > 0$;
(F2) $F(t, 0, 0, t, t, 0) > 0$ for all $t > 0$;
(F3) $F(t, t, 0, 0, t, t) > 0$ for all $t > 0$.

Consider in $\mathbb{R}^6$ the following vectors : $v_1 = (1, 0, 1, 0, 0, 1)$, $v_2 = (1, 0, 0, 1, 1, 0)$ and $v_3 = (1, 1, 0, 0, 1, 1)$. Then the conditions (F1), (F2) and (F3) can be written in a unified form as: for each $k \in \{1, 2, 3\}$,

$$F(tv_k) > 0 \text{ for all } t > 0.$$

Below are some examples of functions belonging to the class $\mathcal{F}$ introduced by Ali and Imdad.

Example 2.2. $F_1(t_1, \ldots, t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, *where* $a, b, c, d, e \geq 0$ *and* $a + b + c + d + e < 1$.

Indeed, $F_1(tv_1) = t(1 - b - e)$, $F_1(tv_2) = t(1 - c - d)$ and $F_1(tv_3) = t(1 - a - d - e)$ are positive for all $t > 0$.

Example 2.3. [2] $F_2(t_1, \ldots, t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\}$, *where* $k \in [0, 1)$.

Example 2.4. $F_3(t_1, \ldots, t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\right\}$, *where* $k \in [0, 1)$.

For each $k \in \{1, 2, 3\}$, $F_2(tv_k) = F_3(tv_k) = t(1-k) > 0$ for all $t > 0$.

Example 2.5. $F_4(t_1, \ldots, t_6) = t_1 - at_2 - b \max\{t_3, t_4\} - c \max\{t_5, t_6\}$, *where* $a, b, c \geq 0$ *and* $a + b + c < 1$.

For each $k \in \{1, 2\}$, $F_4(tv_k) = t(1-b-c)$, while $F_4(tv_3) = t(1-a-c)$.

Example 2.6. $F_5(t_1, \ldots, t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1 - \alpha)(at_5 + bt_6)$, *where* $\alpha \in [0, 1)$ *and* $a, b \geq 0$ *with* $a + 2b < 1$.

Indeed, $F_5(tv_1) = t(1 - \alpha)(1 - b)$, $F_5(tv_2) = t(1 - \alpha)(1 - a)$ and $F_5(tv_3) = t(1 - \alpha)(1 - a - b)$ are positive for all $t > 0$.

Example 2.7. $F_6(t_1, \ldots, t_6) = t_1 - at_2 - b\frac{t_5+t_6}{1+t_3+t_4}$, *where* $a, b \geq 0$ *and* $a + b < 1$.

For each $k \in \{1, 2\}$, $F_6(tv_k) = t(1-\frac{b}{1+t})$, while $F_4(tv_3) = t(1-a-2b)$.

Example 2.8. $F_7(t_1, \ldots t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, *where* $a, b, c \geq 0$ *and* $\max\{a + b, c\} < 1$.

Here $F_7(tv_1) = t(1 - \max\{b, c\})$, $F_7(tv_2) = t(1 - \max\{a, c\})$ and $F_7(tv_3) = t(1 - \max\{a + b, c\})\, t(1 - \max\{a + b, c\})$.

Example 2.9. $F_8(t_1, \ldots, t_6) = t_1 - at_2 - b(t_3 + t_4) - c\min\{t_5, t_6\}$, *where* $a, b, c \geq 0$ *and* $\max\{a + c, b\} < 1$.

For each $k \in \{1, 2\}$, $F_8(tv_k) = t(1 - b)$, while $F_8(tv_3) = t(1 - a - c)$.
For other examples see [2] and [14].

3. Main results

In the following, $\mathcal{F}$ is the family of functions given by Definition 2.1. For simplicity, for every self-mapping f of X we denote $f(x)$ by fx.

Theorem 3.1. *Let* (X, d) *be a semimetric space and* A, S, B, T *be self-mappings of* X *such that for all* $x, y \in X$ *the following inequality holds*

$$F\left(\begin{array}{lll} \psi(d(Ax, By)), & \psi(d(Sx, Ty)), & \psi(d(Ax, Sx)), \\ \psi(d(By, Ty)), & \psi(d(By, Sx)), & \psi(d(Ax, Ty)) \end{array}\right) \leq 0 \tag{3.1}$$

for some $F \in \mathcal{F}$ *and some almost altering distance function* ψ.

If there exist $u, v \in X$ *such that* $Au = Su$ *and* $Bv = Tv$, *then* $Au = Bv$ *and* $z = Au = Su = Bv = Tv$ *is the unique coincidence value of* A *and* S *and the unique coincidence value of* B *and* T.

Proof. First we prove that $Au = Bv$. It follows that $z := Au = Su = Bv = Tv$ is a coincidence value of A and S and also a coincidence value of B and T.

Substituting $x = u$ and $y = v$ in (3.1), we obtain

$$F\left(\begin{array}{lll} \psi(d(Au, Bv)), & \psi(d(Su, Tv)), & \psi(d(Au, Su)), \\ \psi(d(Bv, Tv)), & \psi(d(Bv, Su)), & \psi(d(Au, Tv)) \end{array}\right) \leq 0.$$

As $Au = Su$ and $Bv = Tv$, the inequality above is equivalent to

$$F(\psi(d(Au, Bv)), \psi(d(Au, Bv)), 0, 0, \psi(d(Au, Bv)), \psi(d(Au, Bv))) \leq 0. \tag{3.2}$$

Here we used the axioms (S1) (sufficiency) and (S2) of the semimetric d.

If $Au \neq Bv$, then the axiom (S1) (necessity) shows that $d(Au, Bv) > 0$, therefore $\psi(d(Au, Bv)) > 0$, according to condition (A2) (necessity) from the definition of an almost altering distance function. In this case, (3.2) contradicts condition (F3) from the definition of $\mathcal{F}$. We proved that $Au = Bv$.

Assume that there exists a coincidence value $t \in X$ of A and S, i.e. there exists $w \in X$ such that $t = Aw = Sw$. Setting $x = u$ and $y = v$

in (3.1), we get

$$F(\psi(d(t,z)),\psi(d(t,z)),0,0,\psi(d(t,z)),\psi(d(t,z)))\leq 0.$$

If $t\neq z$, then we conclude as above that the latter inequality contradicts (F3). It follows that $t=z$, hence A and S have the unique coincidence value z.

Similarly, z is the unique coincidence value of B and T. Indeed, assume that there exists $p\in X$ such that $Bp=Tp$ and denote $q:=Bp=Tp$. By (3.1), for $x=u$ and $y=p$ we get

$$F(\psi(d(z,q)),\psi(d(z,q)),0,0,\psi(d(z,q)),\psi(d(z,q)))\leq 0.$$

If $q\neq z$, then the latter inequality contradicts (F3). It follows that $q=z$, as claimed. $\square$

Remark 3.2. *In Theorem 3.1, one may relax the assumption that $F\in\mathcal{F}$; it is enough to assume that $F:\mathbb{R}_+^6\longrightarrow\mathbb{R}$ satisfies condition (F3) from the definition 2.1 of the class $\mathcal{F}$.*

We recall that the existence of a unique coincidence value ensures the existence of a unique common fixed point, for a pair of weakly compatible self-mappings of an arbitrary non-empty set.

Lemma 3.3. [1] *Let f and g be weakly compatible self-mappings of a non-empty set X. If f and g have a unique coincidence value $w=fx=gx$, where $x\in X$, then w is the unique common fixed point of f and g.*

We will consider two pair of mappings satisfying the generalized contractive condition (3.1) from Theorem 3.1, assumming in addition that each pair is weakly compatible and that the first pair satisfies the common limit range property with respect to a mapping from the second pair, see Definition 1.6. A common fixed point result is obtained, as follows.

Theorem 3.4. *Let (X,d) be a semimetric space, where d is $1-$continuous and satisfies condition (H.E.) from Definition 1.3. Let A,S,B,T be self-mappings of X satisfying (3.1) for all $x,y\in X$, where $F\in\mathcal{F}$ and ψ is an almost altering distance function.*

If the pair (A,S) satisfies the common limit range property with respect to T, then

(1) A and S have at least one coincidence point and

(2) B and T have at least one coincidence point.

Moreover, if both (A,S) and (B,T) are pairs of weakly compatible mappings, then

(3) A, S, B, T have a unique common fixed point.

Proof. Since (A, S) satisfies the common limit range property with respect to T, there exists a sequence $(x_n)_{n\geq 1}$ in X such that $\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} S(x_n) =: z$, where $z \in S(X) \cap T(X)$.

Since $z \in T(X)$, there exists $u \in X$ such that $z = Tu$. By (3.1) for $x = x_n$ and $y = u$, we obtain

$$F\left(\begin{array}{ccc} \psi(d(Ax_n, Bu)), & \psi(d(Sx_n, Tu)), & \psi(d(Ax_n, Sx_n)), \\ \psi(d(Bu, Tu)), & \psi(d(Bu, Sx_n)), & \psi(d(Ax_n, Tu)) \end{array}\right) \leq 0. \tag{3.3}$$

Having $\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} S(x_n) =: z$, the 1-continuity of d and the continuity of ψ imply $\lim_{n\to\infty} \psi(d(Ax_n, Bu)) = \psi(d(z, Bu))$, $\lim_{n\to\infty} \psi(d(Sx_n, Tu)) = \psi(d(z, Tu)) = 0$, $\lim_{n\to\infty} \psi(d(Bu, Sx_n)) = \psi(d(Bu, z))$ and $\lim_{n\to\infty} \psi(d(Ax_n, Tu)) = \psi(d(z, Tu)) = 0$. In addition, $\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} S(x_n) =: z$, the property (H.E.) of d and the continuity of ψ imply $\lim_{n\to\infty} \psi(d(Ax_n, Sx_n)) = \psi(0) = 0$. Denote $t = \psi(d(z, Bu))$.

Applying $\liminf_{n\to\infty}$ in (3.3), taking into account the above limits and the lower semicontinuity of F, we obtain

$$F(t, 0, 0, t, t, 0) \leq 0. \tag{3.4}$$

If $Bu \neq z$, then $t > 0$, by axiom (S1) (necessity) from Definition 1.1 and axiom (A2) (necessity) from Definition 1.8. If $t > 0$, then (3.4) contradicts the assumption that F satisfies condition (F2). This contradiction shows that

$$Bu = z = Tu,$$

therefore u is a coincidence point of the pair (B, T).

On the other hand, $z \in S(X)$ implies the existence of a point $v \in X$ such that $z = Sv$. Setting $x = v$ and $y = u$ in (3.1), we get

$$F\left(\begin{array}{ccc} \psi(d(Av, Bu)), & \psi(d(Sv, Tu)), & \psi(d(Av, Sv)), \\ \psi(d(Bu, Tu)), & \psi(d(Bu, Sv)), & \psi(d(Av, Tu)) \end{array}\right) \leq 0.$$

i.e.

$$F(\psi(d(Av, z)), 0, \psi(d(Av, z)), 0, 0, \psi(d(Av, z))) \leq 0. \tag{3.5}$$

If $Av \neq z$, then $\psi(d(Av, z)) > 0$ and (3.5) contradict the assumption that F satisfies condition (F1). It follows that $Av = z = Sv$, therefore, v is a coincidence point of the pair (A, S).

Note that $z = Av = Sv = Bu = Tu$ is a coincidence value for each of the pairs (A, S) and (B, T). As F satisfies condition (F3), by Theorem 3.1 it follows that z is the unique coincidence value of A and S and the unique coincidence value of B and T.

Finally, assume that both (A, S) and (B, T) are pairs of weakly compatible mappings. As $z = Av = Sv = Bu = Tu$, applying Lemma 3.3 for the pairs (A, S) and (B, T) we find that z is the unique common fixed point of A, S, B and T. $\square$

Remark 3.5. *In Theorem 3.4, in order to obtain the conclusions (1) and (2) it suffices to assume that* $F : \mathbb{R}_+^6 \longrightarrow \mathbb{R}$ *satisfies the conditions (F1) and (F2) from the Definition 2.1 of the class* $\mathcal{F}$.

Considering the special case where the almost altering function ψ is the identity function, by Theorem 3.4 and the above we obtain the following result, which generalizes Theorem 1 from [16].

Corollary 3.6. *Let* (X, d) *be a semimetric space, where* d *is* $1-$*continuous and satisfies condition (H.E.) from Definition 1.3. Let* A, S, B, T *be self-mappings of* X *such that for all* $x, y \in X$ *the following inequality holds*

(3.6)

$$F(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(By, Sx), d(Ax, Ty)) \leq 0$$

for some $F : \mathbb{R}_+^6 \longrightarrow \mathbb{R}$ *satisfying conditions (F1) and (F2) from Definition 2.1.*

If the pair (A, S) *satisfies the common limit range property with respect to* T*, then*

(1) A *and* S *have at least one coincidence point and*

(2) B *and* T *have at least one coincidence point.*

Moreover, if both (A, S) *and* (B, T) *are pairs of weakly compatible mappings and if* F *satisfies condition (F3) from Definition 2.1, then*

(3) A, S, B, T *have a unique common fixed point.*

The following examples show that the assumptions of Corollary 3.6 can be satisfied simultaneously.

Example 3.7. *Consider the symmetric space* (X, d) *from Example 1.7, with* $X = \mathbb{R}_+$ *and* $d(x, y) = (x - y)^2$ *for* $x, y \in X$. *We check that* d *is* $1-$*continuous and satisfies condition (H.E.).*

a) Assume that $\lim_{n\to\infty} d(x_n, x) = 0$*, which is equivalent to* $\lim_{n\to\infty} |x_n - x| = 0$. *Let* $y \in X$. *As* $|d(x_n, y) - d(x, y)| = |x_n - x| \, |x_n + x - 2y|$*, for every* $n \geq 1$ *and the sequence* $(x_n)_{n\geq 1}$ *is* *bounded in* $(\mathbb{R}, |\cdot|)$, *it follows that* $\lim\limits_{n\to\infty} d(x_n, y) = d(x, y)$. *We proved that* d *is* $1-$*continuous.*

b) Assume that $\lim\limits_{n\to\infty} d(x_n, x) = 0$ *and* $\lim\limits_{n\to\infty} d(y_n, x) = 0$. *We will prove that* $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$, *which shows that* d *satisfies the condition (H.E.). As* $\lim\limits_{n\to\infty} d(x_n, x) = 0$ *implies* $\lim\limits_{n\to\infty} |x_n - x| = 0$ *and* $\lim\limits_{n\to\infty} d(y_n, x) = 0$ *implies* $\lim\limits_{n\to\infty} |y_n - x| = 0$, *by the usual triangle inequality in* $\mathbb{R}$ *it follows that* $\lim\limits_{n\to\infty} |x_n - y_n| = 0$. *Then* $\lim\limits_{n\to\infty} d(x_n, y_n) = \lim\limits_{n\to\infty} (x_n - y_n)^2 = 0$, *as required.*

Example 3.8. *Consider the symmetric space* (X, d) *from Example 1.7, with* $X = \mathbb{R}_+$ *and* $d(x, y) = (x - y)^2$ *for* $x, y \in X$. *Let* A, S, B, T *be self-mappings of* X *defined by* $Ax = ax$, $Sx = sx$, $Bx = bx$ *and* $Tx = tx$, *for all* $x \in X$, *for some distinct positive constants* a, s, b, t. *Then* $S(X) = [0, s]$ *and* $T(X) = [0, d]$, *hence* $S(X) \cap T(X) = [0, \min\{s, t\}]$. *We check that the pair* (A, S) *satisfies the common limit range property with respect to* T, *and that* (A, S) *and* (B, T) *are pairs of weakly compatible mappings.*

c) Assume that $(z_n)_{n\geq 1}$ *is a sequence in* X, *with* $\lim\limits_{n\to\infty} z_n = 0$. *Then* $\lim\limits_{n\to\infty} A(z_n) = \lim\limits_{n\to\infty} S(z_n) = 0 \in S(X) \cap T(X)$, *therefore* (A, S) *satisfies the common limit range property with respect to* T.

d) $Ax = Sx$ *if and only if* $x = 0$ *and* $A(S(0)) = S(A(0)) = 0$, *therefore* A *and* S *are weakly compatible. Similarly,* $Bx = Tx$ *if and only if* $x = 0$ *and* $B(T(0)) = T(B(0)) = 0$, *therefore* B *and* T *are weakly compatible.*

Now we check that the following condition is satisfied:

e) there exists $F \in \mathcal{F}$ *such that (3.6) holds for all* $x, y \in X$.

Note that $d(Ax, By) = (ax - by)^2$, $d(Sx, Ty) = (sx - ty)^2$, $d(Ax, Sx) = (a-s)^2x^2$, $d(By, Ty) = (b-t)^2y^2$, $d(By, Sx) = (sx-by)^2$, $d(Ax, Ty) = (ax - ty)^2$.

We look for some positive constants a, s, b, t *and* c *such that (3.6) is satisfied with* $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k\max\{t_2, t_3, t_4, t_5, t_6\}$, *for some* $k \in [0, 1)$. *By Example 2.3, it is known that* $F \in \mathcal{F}$.

It suffices to require that for all $x, y \in X$ *one of the following inequalities holds:*

$$\begin{aligned}(3.7)\qquad (ax - by)^2 &\leq k(sx - ty)^2;\\ (ax - by)^2 &\leq k(a - s)^2x^2;\ (ax - by)^2 \leq k(b - t)^2y^2;\\ (ax - by)^2 &\leq k(sx - by)^2;\ (ax - by)^2 \leq k(ax - ty)^2.\end{aligned}$$

In general, assuming that a, b, p, q are not all zero, then $(ax - by)^2 \leq k(px - qy)^2$ for all $x, y \in X = \mathbb{R}_+$ if and only if there exists $\lambda \in \mathbb{R}$ with $|\lambda| \leq \sqrt{k}$ such that $a = \lambda p$ and $b = \lambda q$.

Since we require $k \in [0, 1)$ and $a, s, b, t > 0$, only the first case from (3.7) is possible, namely $(ax-by)^2 \leq k(sx-ty)^2$ for all $x, y \in X = \mathbb{R}_+$ and this condition is satisfied if and only if there exists $\lambda \in \mathbb{R}$ with $0 < \lambda \leq \sqrt{k}$ such that $a = \lambda s$ and $b = \lambda t$.

Taking $k \in (0, 1)$, two distinct positive numbers s, t and $a = s\sqrt{k}$ and $b = t\sqrt{k}$, we obtain that the following inequality is satisfied for all $x, y \in X = \mathbb{R}_+$:

$$\begin{aligned} d(Ax, By) &= kd(Sx, Ty) \\ &\leq k \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(By, Sx), d(Ax, Ty)\} \end{aligned}$$

4. Applications

4.1. **Fixed points for mappings satisfying contractive conditions of integral type.** In [9], Branciari proved the following theorem, a strong generalization of Banach contraction principle, which opened the way for the study of fixed points for mappings satisfying contractive conditions of integral type.

Theorem 4.1. *Let (X, d) be a complete metric space, $c \in (0, 1)$ and let $f : X \to X$ be a mapping such that, for each $x, y \in X$,*

$$\int_0^{d(fx,fy)} h(t)dt \leq c \int_0^{d(x,y)} h(t)dt, \tag{4.1}$$

where $h : [0, +\infty) \to [0, +\infty]$ is a Lebesgue-measurable function which is summable (i.e., with finite integral) on each compact subset of $[0, +\infty)$, such that $\int_0^{\varepsilon} h(t)dt > 0$ for all $\varepsilon > 0$. Then f has a unique fixed point $z \in X$ such that $\lim_{n\to\infty} f^n x = z$ for all $x \in X$.

Lemma 4.2. [28]*Let $h : [0, +\infty) \to [0, +\infty]$ be a Lebesgue-measurable function that is summable on each compact subset of $[0, +\infty)$, such that $\int_0^{\varepsilon} h(t)dt > 0$ for all $\varepsilon > 0$. Define $\psi(t) = \int_0^{t} h(\tau)d\tau$, $t \in [0, +\infty)$. Then ψ is an altering distance.*

Theorem 4.3. *Let (X, d) be a semimetric space, where d is $1-$continuous and satisfies the condition (H.E.) from Definition 1.3.*

Let A, S, B, T be self-mappings of X satisfying for all $x, y \in X$ the following condition, for some $F \in \mathcal{F}$ and some function $h : [0, +\infty) \to [0, +\infty]$ which is Lebesgue-measurable and summable on each compact subset of $[0, +\infty)$:

$$(4.2) \qquad F\left(\begin{array}{ccc} \int\limits_0^{d(Ax,By)} h(t)dt, & \int\limits_0^{d(Sx,Ty)} h(t)dt, & \int\limits_0^{d(Ax,Sx)} h(t)dt, \\ \int\limits_0^{d(By,Ty)} h(t)dt, & \int\limits_0^{d(By,Sx)} h(t)dt, & \int\limits_0^{d(Ax,Ty)} h(t)dt \end{array}\right) \leq 0$$

If the pair (A, S) satisfies the common limit range property with respect to T, then

Theorem 4.4. *(1) A and S have at least one coincidence point and (2) B and T have at least one coincidence point.*

Moreover, if both (A, S) and (B, T) are pairs of weakly compatible mappings, then

(3) A, S, B, T have a unique common fixed point.

Proof. Define $\psi(t) = \int\limits_0^t h(\tau)d\tau$, $t \in [0, +\infty)$. According to Lemma 4.2, ψ is an altering distance, hence ψ is an almost altering distance. Now condition (4.2) writes as follows, for all $x, y \in X$:

$$F\left(\begin{array}{ccc} \psi(d(Ax, By)), & \psi(d(Sx, Ty)), & \psi(d(Ax, Sx)), \\ \psi(d(By, Ty)), & \psi(d(By, Sx)), & \psi(d(Ax, Ty)) \end{array}\right) \leq 0.$$

The above inequality is (3.1) from Theorem 3.1 and Theorem 3.4. Applying Theorem 3.4, all the claims follow. □

By Theorem 4.3 and Example 2.3, we obtain the following

Corollary 4.5. *Let (X, d) be a semimetric space, where d is $1-$continuous and satisfies condition (H.E.) from Definition 1.3. Let A, S, B, T be self-mappings of X satisfying the following condition, for some $F \in \mathcal{F}$, a constant $k \in [0, 1)$ and some function $h : [0, +\infty) \to [0, +\infty]$, which is Lebesgue-measurable and summable on each compact subset of $[0, +\infty)$: for all $x, y \in X$,*

(4.3)

$$\int\limits_0^{d(Ax,By)} h(t)dt \leq k \max\left\{\begin{array}{ccc} \int\limits_0^{d(Sx,Ty)} h(t)dt, & \int\limits_0^{d(Ax,Sx)} h(t)dt, & \int\limits_0^{d(By,Ty)} h(t)dt, \\ \int\limits_0^{d(By,Sx)} h(t)dt, & \int\limits_0^{d(Ax,Ty)} h(t)dt & \end{array}\right\}$$

If the pair (A, S) satisfies the common limit range property with respect to T, then A and S have at least one coincidence point and B and T have at least one coincidence point. Moreover, if both (A, S) and (B, T) are pairs of weakly compatible mappings, then A, S, B, T have a unique common fixed point.

4.2. **Fixed point results in semimetric spaces via C-class functions.** Below we give a lemma that provides some general examples of $C-$class functions, a concept introduced by Ansari [4], see Definition 1.9. For particular examples, see [4], [5], [8].

Lemma 4.6. *Let $G : [0, \infty) \times [0, \infty) \to \mathbb{R}$. Then G is a $C-$class function in each of the following cases:*

1) $G(s, t) = f_2(s) - f_1(t)$, where f_1, $f_2 : [0, \infty) \to \mathbb{R}$ are continuous functions, with $f_1(t) \geq 0$ for all $t \in [0, \infty)$ and $f_2(s) \leq s$ for all $s \in [0, \infty)$, such that: $f_1(t) = 0$ implies $t = 0$ or $f_2(s) = s$ implies $s = 0$.

(2) $G(s, t) = f_3(s)f_4(t)$, where f_3, $f_4 : [0, \infty) \to [0, \infty)$ are continuous functions, with $f_4(t) \leq 1$ for all $t \in [0, \infty)$ and $f_3(s) \leq s$ for all $s \in [0, \infty)$, such that: $f_4(t) = 1$ implies $t = 0$ or $f_3(s) = s$ implies $s = 0$.

Proof. 1) The continuity of $G : [0, \infty) \times [0, \infty) \to \mathbb{R}$ follows from the continuity of f_1, $f_2 : [0, \infty) \to \mathbb{R}$ and the continuity of the projections p_1, $p_2 : [0, \infty) \times [0, \infty) \to [0, \infty)$ with $p_1(s, t) = s$ and $p_2(s, t) = t$ for all $(s, t) \in [0, \infty) \times [0, \infty)$.
We have $G(s, t) = f_2(s) - f_1(t) \leq f_2(s) \leq s$ for all $(s, t) \in [0, \infty) \times [0, \infty)$, therefore, the condition (G1) from Definition 1.9 is satisfied. If $G(s, t) = s$, then the above inequalities hold as equalities, hence $f_1(t) = 0$ and $f_2(s) = s$. If $f_1(t) = 0$ implies $t = 0$, then we obtain $t = 0$. If $f_2(s) = s$ implies $s = 0$, then we get $s = 0$. In any case, it follows that $s = 0$ or $t = 0$.

2) The continuity of $G : [0, \infty) \times [0, \infty) \to \mathbb{R}$ follows from the continuity of f_3, $f_4 : [0, \infty) \to \mathbb{R}$ and the continuity of the projections p_1, $p_2 : [0, \infty) \times [0, \infty) \to [0, \infty)$.

The condition (G1) from Definition 1.9 is satisfied, since $G(s, t) = f_3(s)f_4(t) \leq f_3(s) \leq s$. Assume now that $G(s, t) = s$. Then the latter inequalities hold as equalities, so (i) $f_3(s) = s$ and (ii) $f_3(s)\left(f_4(t) - 1\right) = 0$. If it is assumed that $f_3(s) = s$ implies $s = 0$, then by (i) we get $s = 0$. In the case where $f_3(s) = s \neq 0$, the above assumption on f_3 does not hold, therefore $f_4(t) = 1$ implies $t = 0$, whence using (ii) we get $t = 0$. In any case, it follows that $s = 0$ or $t = 0$, therefore G satisfies condition (G2). $\square$

In the following examples, $G : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is a $C-$class function, ψ is an almost altering distance function, and $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(t) > 0$ for all $t > 0$. In all these examples, it turns out that the function $H_k : \mathbb{R}_+^6 \longrightarrow \mathbb{R}$ belongs to the class $\mathcal{F}$. So, using $C-$class function and almost altering distance functions many examples of functions belonging to the class $\mathcal{F}$ can be obtained.

For brevity, denote $v_1 = (1, 0, 1, 0, 0, 1)$, $v_2 = (1, 0, 0, 1, 1, 0)$ and $v_3 = (1, 1, 0, 0, 1, 1)$.

Example 4.7.

$$\begin{aligned} H_1(t_1, \ldots, t_6) &= \psi(t_1) - G(\psi(f_1(t_1, \ldots, t_6)), \varphi(f_1(t_1, \ldots, t_6))), \\ \textit{where } f_1(t_1, \ldots, t_6) &= \max\{t_2, t_3, t_4, t_5, t_6\} \end{aligned}$$

Example 4.8.

$$\begin{aligned} H_2(t_1, \ldots, t_6) &= \psi(t_1) - G(\psi(f_2(t_1, \ldots, t_6)), \varphi(f_2(t_1, \ldots, t_6))), \\ \textit{where } f_2(t_1, \ldots, t_6) &= \max\{t_2, t_3, t_4, (t_5 + t_6)/2\}. \end{aligned}$$

Example 4.9.

$$H_3(t_1, \ldots, t_6) = \psi(t_1) - G(\psi(f_1(t_1, \ldots, t_6)), \varphi(f_2(t_1, \ldots, t_6))),$$

where $f_1(t_1, \ldots, t_6) = \max\{t_2, t_3, t_4, t_5, t_6\}$ *and* $f_2(t_1, \ldots, t_6) = \max\{t_2, t_3, t_4, (t_5 + t_6)/2\}$.

Example 4.10.

$$H_3(t_1, \ldots, t_6) = \psi(t_1) - G(\psi(f_2(t_1, \ldots, t_6)), \varphi(f_1(t_1, \ldots, t_6))),$$

where $f_1(t_1, \ldots, t_6) = \max\{t_2, t_3, t_4, t_5, t_6\}$ *and* $f_2(t_1, \ldots, t_6) = \max\{t_2, t_3, t_4, (t_5 + t_6)/2\}$.

Example 4.11.

$$\begin{aligned} H_5(t_1, \ldots, t_6) &= \psi(t_1) - G(\psi(f_3(t_1, \ldots, t_6)), \varphi(f_1(t_1, \ldots, t_6))), \\ \textit{where } f_3(t_1, \ldots, t_6) &= \max\left\{\sqrt{t_2 t_5}, \sqrt{t_3 t_6}, \sqrt{t_4 t_5}, \sqrt{t_5 t_6}\right\} \end{aligned}$$

and $f_1(t_1, \ldots, t_6) = \max\{t_2, t_3, t_4, t_5, t_6\}$.

Example 4.12.

$$\begin{aligned} H_6(t_1, \ldots, t_6) &= \psi(t_1) - G(\psi(f_4(t_1, \ldots, t_6)), \varphi(f_4(t_1, \ldots, t_6))), \\ \textit{where } f_4(t_1, \ldots, t_6) &= \max\left\{t_2, t_3, t_4, \frac{t_3 t_4}{1 + t_2}, \frac{t_5 t_6}{1 + t_1}\right\}. \end{aligned}$$

We will check that the functions given in the above examples are in the class $\mathcal{F}$.
Let $i \in \{1,2,3,4,5,6\}$. For each $k \in \{1,2,3\}$, we have $H_i(tv_k) = \psi(t) - G(\psi(t), \varphi(t))$. By condition (G1) from Definition 1.9 $H_i(tv_k) \geq 0$ for all $t \geq 0$. According to condition (G2) from Definition 1.9, $H_i(tv_k) = 0$ implies $\psi(t) = 0$ or $\varphi(t) = 0$, hence $t = 0$, by condition (A2) from Definition 1.8 and the assumption $\varphi(t) > 0$ for all $t > 0$. Then $H_i(tv_k) > 0$ for all $t > 0$ and all $k \in \{1,2,3\}$, therefore H_i belongs to the class $\mathcal{F}$, according to Definition 2.1.

Example 4.13.

$$\begin{aligned} H_7\left(t_1,\ldots,t_6\right) &= \psi(t_1) - G\left(\psi(f_5\left(t_1,\ldots,t_6\right)), \varphi(f_2\left(t_1,\ldots,t_6\right))\right), \\ \textit{where } f_5\left(t_1,\ldots,t_6\right) &= \sqrt{t_2t_3} + \sqrt{t_3t_4} + \sqrt{t_3t_6} + \sqrt{t_5t_6} \\ \textit{and } f_2\left(t_1,\ldots,t_6\right) &= \max\left\{t_2, t_3, t_4, (t_5+t_6)/2\right\} \end{aligned}$$

For $k \in \{1,3\}$, we have $H_7(tv_k) = \psi(t) - G(\psi(t), \varphi(t))$ and, as above, we see that $H_7(tv_k) > 0$ for all $t > 0$. In addition, $H_7(tv_2) = \psi(t) - G(\psi(0), \varphi(t)) = \psi(t) - G(0, \varphi(t))$, but $G(0, \varphi(t)) \leq 0$ for all $t \geq 0$ and $\psi(t) > 0$ for all $t > 0$, hence $H_7(tv_2) > 0$ for all $t > 0$.

In the following example we will assume that ψ is an *altering distance function*, as monotonicity is needed.

Example 4.14.

$$\begin{aligned} H_8\left(t_1,\ldots,t_6\right) &= \psi(t_1) - G\left(\psi(f_6\left(t_1,\ldots,t_6\right)), \varphi(f_1\left(t_1,\ldots,t_6\right))\right), \\ \textit{where } f_6\left(t_1,\ldots,t_6\right) &= \frac{\sqrt{t_3t_6} + \sqrt{t_4t_5} + \sqrt{t_5t_6}}{1+t_3} \\ \textit{and } f_1\left(t_1,\ldots,t_6\right) &= \max\left\{t_2, t_3, t_4, t_5, t_6\right\}. \end{aligned}$$

For $k \in \{2,3\}$, we have $H_8(tv_k) = \psi(t) - G(\psi(t), \varphi(t))$, hence $H_8(tv_k) > 0$ for all $t > 0$. In addition, $H_8(tv_1) = \psi(t) - G\left(\psi\left(\frac{t}{1+t}\right), \varphi(t)\right) \geq \psi(t) - \psi\left(\frac{t}{1+t}\right)$. As we assumed that ψ is non-decreasing and $\frac{t}{1+t} \leq t$ for all $t \geq 0$, it follows that $H_8(tv_1) \geq \psi(t) - \psi\left(\frac{t}{1+t}\right) \geq 0$ for all $t \geq 0$. Moreover, if $H_8(tv_1) = 0$, then $\psi\left(\frac{t}{1+t}\right) = 0$ or $\varphi(t) = 0$, hence $t = 0$.

Theorem 4.15. *Let (X,d) be a semimetric space, where d is $1-$continuous and satisfies condition (H.E.) from Definition 1.3. Assume that $G : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is a $C-$class function, ψ is an almost altering distance function and $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) > 0$ for all $t > 0$.*
Let $f_1, f_2 : [0,\infty)^5 \to [0,\infty)$ such that for each $i \in \{1,2\}$, all $t \geq 0$

and all $(t_2, t_3, t_4, t_5, t_6) \in \{(0, t, 0, 0, t); (0, 0, t, t, 0); (t, 0, 0, t, t)\}$, *we have* $f_i(t_2, t_3, t_4, t_5, t_6) = t$.
Define $\omega : [0, \infty)^5 \to [0, \infty) \times [0, \infty)$ *by* $\omega = (\psi \circ f_1, \varphi \circ f_2)$.

Let A, S, B, T *be self-mappings of* X *such that for all* $x, y \in X$ *the following inequality holds*

(4.4)
$$\psi(d(Ax, By)) \leq G(\omega(d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(By, Sx), d(Ax, Ty))).$$

If the pair (A, S) *satisfies the common limit range property with respect to* T*, then*

(1) A *and* S *have at least one coincidence point and*

(2) B *and* T *have at least one coincidence point.*

Moreover, if both (A, S) *and* (B, T) *are pairs of weakly compatible mappings, then*

(3) A, S, B, T *have a unique common fixed point.*

Proof. Define $H : [0, \infty)^6 \to [0, \infty)$ by

$$\begin{aligned} H(t_1, t_2, t_3, t_4, t_5, t_6) &= \psi(t_1) \\ &\quad - G\left(\psi\left(f_1\left(t_2, t_3, t_4, t_5, t_6\right)\right), \varphi(f_2\left(t_2, t_3, t_4, t_5, t_6\right))\right). \end{aligned}$$

For all $t \geq 0$ and for all

$(t_2, t_3, t_4, t_5, t_6) \in \{(0, t, 0, 0, t); (0, 0, t, t, 0); (t, 0, 0, t, t)\}$, we can see by calculations that $G\left(\psi\left(f_1\left(t_2, t_3, t_4, t_5, t_6\right)\right), \varphi(f_2\left(t_2, t_3, t_4, t_5, t_6\right))\right) = G\left(\psi\left(t\right), \varphi(t)\right)$.

Consider in $\mathbb{R}^6$ the following vectors : $v_1 = (1, 0, 1, 0, 0, 1)$, $v_2 = (1, 0, 0, 1, 1, 0)$ and $v_3 = (1, 1, 0, 0, 1, 1)$. For each $k \in \{1, 2, 3\}$ and all $t \geq 0$,

$$H(tv_k) = \psi(t) - G\left(\psi\left(t\right), \varphi(t)\right).$$

By condition (G1) from Definition 1.9 $H(tv_k) \geq 0$ for all $t \geq 0$. According to condition (G2) from Definition 1.9, $H(tv_k) = 0$ implies $\psi(t) = 0$ or $\varphi(t) = 0$, hence $t = 0$, by condition (A2) from Definition 1.8 and the assumption $\varphi(t) > 0$ for all $t > 0$.
Then $H(tv_k) > 0$ for all $t > 0$ and all $k \in \{1, 2, 3\}$, therefore H belongs to the class $\mathcal{F}$, according to Definition 2.1.

The contractive condition (4.4) can be written as: for all $x, y \in X$,

$$H(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(By, Sx), d(Ax, Ty)) \leq 0,$$

Applying Corollary 3.6 with $F := H$ we obtain the claim. $\square$

References

[1] M. Abbas and B. E, Rhoades, **Common fixed point results for noncommuting mappings without continuity in generalized metric spaces**, Appl. Math. Comput, 215 (2009), 262-269.

[2] J. Ali and M. Imdad, **An implicit function implies several contraction conditions**, Sarajevo J. Math., 17 (4) (2008), 269-285.

[3] A. Aliouche, **A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type**, *J. Math. Anal. Appl.* **322**(2006), 796-802.

[4] A. H. Ansari, **Note on $\varphi - \psi$-contractive type mappings and related fixed point**, The 2nd Regional Conference on Mathematics and Applications, PNU, 2014, pp. 377-380.

[5] A. H. Ansari and S. Beloul, $C-$**class functions on common fixed points for mappings satisfying linear contractive condition**, Survey Math. Appl. 12 (2017), 35-49.

[6] A. H. Ansari and P. S. Kumari, **Fixed point theorems via $C-$class functions in symmetric spaces**, Commun. Optim. Theory, 2017, article ID 7.

[7] A. H. Ansari and A. Razani, **Some fixed point theorems for C-class functions in $b-$metric spaces**, Sahand Communications in Mathematical Analysis, 10 1 (2018): 85-96, doi: 10.22130/scma.2017.28505

[8] S. Beloul and A. H. Ansari, **Common fixed points for weakly subsequentially continuous mappings via new function**, J. Adv. Math. Stud., 10 (1) (2017), 62-73.

[9] A. Branciari, **A fixed point theorem for mappings satisfying a general contractive condition of integral type**, *Internat.J. Math. Math. Sci.* **29**,9 (2002), 531-536.

[10] S. Chauhan and E. Karapinar, Some integral type common fixed point theorems satisfying $\Phi-$**contractive conditions,** Bull. Belg. Math. Soc. Simon Stevin, 21 (2009), 593-612.

[11] K. Chrzaszcz, J. Jachymski and F. Turoboros, **On characterizations and topology of regular semimetric spaces**, Publ. Math. Debrecen, 93/1-2 (2018), 87-105.

[12] F. Galvin and S.D. Shore, **Completeness in semimetric spaces**, Pac. J. Math. 439(1) (1984), 67–76.

[13] M. Imdad, B. D. Pant and S. Chauhan, **Fixed point theorems in Menger spaces using the (CLR_{ST}) property and applications,** Journal of Nonlinear Analysis and Optimization, vol. 3 no. 2 (2012), 225-237.

[14] M. Imdad and S. Chauhan, **Employing common limit range property to prove unified metrical common fixed point theorems**, International Journal of Analysis. (2013) 2013, 10, 763261, https://doi.org/10.1155/2013/763261.

[15] M. Imdad, S. Chauhan and Z. Kadelburg, **Fixed point theorems for mappings with common limit range property satisfying generalized (ψ, φ)-weak contractive conditions**, Mathematical Sciences, 7, 16 (2013), https://doi.org/10.1186/2251-7456-7-16

[16] M. Imdad, A. Sharma and S. Chauhan, **Unifying a multitude of common fixed point theorems in symmetric spaces**, Filomat 28 (6) (2014), 1113-1132.

[17] M. Imdad, S. Chauhan, Z. Kadelburg and C. Vetro, **Fixed point theorems for non-self mappings in symmetric spaces under φ-weak contractive conditions and an application to functional equations in dynamic programming**, Applied Mathematics and Computation 227 (2014), 469-479.

[18] G. Jungck, **Compatible mappings and common fixed points**, *Internat. J. Math. Math. Sci.*, **9** (1986), 771-779.

[19] G Jungck, **Common fixed points for noncontinuous nonself maps on nonnumeric spaces**, Far East J. Math. Sci., **4**(2) (1996), 199-215.

[20] Q. H. Khan, S. Askar, S. Ali and H. Ahmad, **Fixed point theorem in symmetric space employing (c)-comparison functions and binary relation**, European Journal of Pure and Applied Mathematics, vol. 17, no. 1 (2024), 310-323.

[21] M.S. Khan, M.Swaleh and S. Sessa, **Fixed point theorems by altering distances between two points**, *Bull. Austral. Math. Soc.* 30(1984), 1-9.

[22] M. Mocanu and V. Popa, **Some fixed point theorems for mappings satisfying implicit relations in symmetric spaces**, Libertas Mathematica, **28**(2008), 1-13.

[23] H. K. Nashine and Z. Kadelburg, **Common fixed point theorems via common limit range property**, Nonlinear Analysis: Modelling and Control 20, no. 3 (2015), 331-347.

[24] V. Popa, **Fixed point theorems for implicit contractive mappings**, Stud. Cerc. St. Ser. Mat., **7** (1997), 127-133.

[25] V Popa, **Some fixed point theorems for compatible mappings satisfying an implicit relation**, *Demonstratio Math.* **32** (1999), 157-163.

[26] V. Popa, **Fixed point theorems for two pairs of mappings satisfying a new type of common limit range property**, Filomat 31 (11) (2017), 3181-3192.

[27] V.Popa and M. Mocanu, **A new viewpoint in the study of fixed points for mappings satisfying a contractive condition of integral type**, *Bul. Inst. Politehn. Iasi, Sect. Mat. Mec. Teor. Fiz.*, **53** (57), 5 (2007), 269-282.

[28] V.Popa and M. Mocanu, **Altering distance and common fixed points under implit relations**, Hacettepe Journal of Mathematics and Statistics, vol. 38, no.3 (2009), 329-337.

[29] V. Popa and A.-M. Patriciu A.-M., **A general fixed point theorem for a pair of self mappings with common limit range property in G-metric spaces**, Facta Univ., Ser. Math. Inf., 29, 4 (2014), 351-370.

[30] B.E. Rhoades, **Two fixed point theorems for mappings satisfying a general contractive conditions of integral type**, *Intern. J. Math. Math. Sci.* 15(2005), 2359-2364.

[31] W. Sintunavarat and P. Kumam, **Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces**, J. Appl. Math., 2011, Article ID 637958, 14 pages.

[32] W. A. Wilson, **On semi-metric spaces**, American Journal of Mathematics, vol. 53, no. 2 (1931): 361-373.

Valeriu Popa
"Vasile Alecsandri" University of Bacău,
Faculty of Sciences,
Department of Mathematics and Informatics,
Calea Mărăşeşti 157, ROMANIA
e-mail: vpopa@ub.ro

Marcelina Mocanu
"Vasile Alecsandri" University of Bacău,
Faculty of Sciences,
Department of Mathematics and Informatics,
Calea Mărăşeşti 157, ROMANIA
e-mail: mmocanu@ub.ro
ORCID 0000-0002-0192-8180