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GROWTH ESTIMATES BASED ON (α, β, γ)-TYPE FOR THE NEVANLINNA CHARACTERISTIC OF A DIFFERENTIAL POLYNOMIAL GENERATED BY A MEROMORPHIC FUNCTION

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Abstract. The notions of (α, β, γ)-order and (α, β, γ)-type have been introduced by B. Belaïdi and T. Biswas, as tools for the investigation the growth of the solutions of linear differential equations with meromorphic coefficients. In this paper, the notions of (α, β, γ)-order and (α, β, γ)-type are used to estimate the growth of the Nevanlinna characteristic of a differential polynomial generated by a meromorphic function by comparison with the Nevanlinna characteristic of a composition of this function with another meromorphic function, one of these functions being entire function.

1. Introduction

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [7, 8, 10, 11]. We also use the standard notations and definitions of the theory of entire functions which are available in [9] and therefore we do not explain those in details.

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Let f be an entire function defined in the open complex plane $\mathbb{C}$ and $M_f(r) = \max\{|f(z)| : |z| = r\}$. When f is meromorphic, one may introduce another function $T_f(r)$, known as Nevanlinna's characteristic function of f (see [7, p.4]), playing the same role as $M_f(r)$, which is defined as

$$T_f(r) = N_f(r) + m_f(r),$$

wherever the function $N_f(r,a)(\overline{N}_f(r,a))$ known as counting function of a-points (distinct a-points) of meromorphic f is defined as follows:

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + n_f(0,a)\log r$$

$$\left(\overline{N}_f(r,a) = \int_0^r \frac{\overline{n}_f(t,a) - \overline{n}_f(0,a)}{t} dt + \overline{n}_f(0,a)\log r\right),$$

in addition we represent by $n_f(r,a)(\overline{n}_f(r,a))$ the number of a-points (distinct a-points) of f in $|z| \leq r$ and an ∞ -point is a pole of f. In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are symbolized by $N_f(r)$ and $\overline{N}_f(r)$ respectively.

On the other hand, the function $m_f(r,\infty)$ alternatively indicated by $m_f(r)$ known as the proximity function of f is defined as:

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \text{ where}$$

$$\log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0.$$

Also we may employ $m(r, \frac{1}{f-a})$ by $m_f(r,a)$.

Remark 1.1. *If f is entire, then the Nevanlinna's characteristic function $T_f(r)$ of f is defined as*

$$T_f(r) = m_f(r) \textit{ for every } r > 0.$$

Further let $n_0, n_1, n_2, ...n_k$ be nonnegative integers. For a transcendental meromorphic function f, we call the expression $M[f] = f^{n_0}(f^{(1)})^{n_1}(f^{(2)})^{n_2}.......(f^{(k)})^{n_k}$ to be a monomial generated by f. The numbers $\gamma_M = n_0 + n_1 + n_2 + ... + n_k$ and $\Gamma_M = n_0 + 2n_1 + 3n_2 + ... + (k+1)n_k$ are called respectively the degree and weight of the monomial. If $M_1[f]$, $M_2[f]$, ..., $M_n[f]$ denote monomials in f, then

$$Q[f] = a_1 M_1[f] + a_2 M_2[f] + ... + a_n M_n[f],$$

where $a_i \neq 0 (i = 1, 2, ..., n)$ is called a differential polynomial generated by f of degree $\gamma_Q = \max\{\gamma_{M_j} : 1 \leq j \leq n\}$ and weight $\Gamma_Q = \max\{\Gamma_{M_J} : 1 \leq j \leq n\}$. Also we call the numbers $\underline{\gamma_Q} = \min\limits_{1 \leq j \leq s} \gamma_{Mj}$ and k (the order of the highest derivative of f) the lower degree and the order of $Q[f]$ respectively. If $\underline{\gamma_Q} = \gamma_Q$, $Q[f]$ is called a homogeneous differential polynomial.

Now, first of all, let L be the class of continuous non-negative on $(-\infty, +\infty)$ function α for which there exists some real number x_0, depending on α, such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $\in L$ and and there exist some real constants R_0 and c such that $\alpha(a+b) \leq \alpha(a)+\alpha(b)+c$ for all $a, b \geq R_0$ and fixed $c \in (0, +\infty)$. Further we say that $\alpha \in L_2$, if $\alpha \in L$ and $\alpha(x+O(1)) = (1+o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_3$ if $\alpha \in L$ and $\alpha(a+b) \leq \alpha(a) + \alpha(b)$ for all $a, b \geq R_0$, i.e., α is subadditive on $[R_0, +\infty)$. Clearly $L_3 \subset L_1$.

Particularly, if $\alpha \in L_3$ and we fix some $R_0 \geq 0$, then it follows by induction over m that $\alpha(mr) \leq m\alpha(r)$ for all $r \geq R_0$ and all positive integers m. Up to a normalization, subadditivity is implied by concavity. Indeed, if $\alpha(r)$ is concave on $[0, +\infty)$ and satisfies $\alpha(0) \geq 0$, then for $t \in [0, 1]$,

$$\begin{aligned} \alpha(tx) &= \alpha(tx + (1-t) \cdot 0) \\ &\geq t\alpha(x) + (1-t)\alpha(0) \geq t\alpha(x), \end{aligned}$$

so that by choosing $t = \frac{a}{a+b}$ or $t = \frac{b}{a+b}$, we obtain

$$\begin{aligned} \alpha(a+b) &= \frac{a}{a+b}\alpha(a+b) + \frac{b}{a+b}\alpha(a+b) \\ &\leq \alpha\left(\frac{a}{a+b}(a+b)\right) + \alpha\left(\frac{b}{a+b}(a+b)\right) \\ &= \alpha(a) + \alpha(b), \ a, b \geq 0. \end{aligned}$$

If α is a non-decreasing, then $\alpha(r)$ satisfies

$$\alpha(r) \leq \alpha(r + R_0) \leq \alpha(r) + \alpha(R_0)$$

for any $R_0 \geq 0$. This yields that $\alpha(r) \sim \alpha(r + R_0)$ as $r \to +\infty$ which implies the ratio $\alpha(r)/\alpha(r + R_0)$ is equal to 1 as $r \to +\infty$.

Throughout this paper we assume that $\alpha \in L_1$, $\beta \in L_2$, $\gamma \in L_3$.

Heittokangas et al. [6] have introduced a new concept of φ-order of entire and meromorphic functions considering φ as subadditive function. For details one may see [6]. Later on Belaïdi et al. [1] introduced the notions of (α,β,γ)-order and (α,β,γ)-lower order of a meromorphic function f, which are defined as follows:

Definition 1.2. [1] *The (α,β,γ)-order denoted by $\rho_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f is defined as:*

$$\rho_{(\alpha,\beta,\gamma)}[f]=\limsup_{r\to+\infty}\frac{\alpha(\log(T_f(r)))}{\beta\left(\log(\gamma(r))\right)}.$$

Definition 1.3. [1] *The (α,β,γ)-lower order denoted by $\lambda_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f is defined as:*

$$\lambda_{(\alpha,\beta,\gamma)}[f]=\liminf_{r\to+\infty}\frac{\alpha(\log(T_f(r)))}{\beta\left(\log(\gamma(r))\right)}.$$

Belaïdi et al. [2] have also introduced the definition of the growth indicator, called (α,β,γ) type of a meromorphic function f in the following way:

Definition 1.4. [2] *The (α,β,γ) type denoted by $\sigma_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f having finite positive (α,β,γ) order $\left(0<\rho_{(\alpha,\beta,\gamma)}[f]<+\infty\right)$ is defined as :*

$$\sigma_{(\alpha,\beta,\gamma)}[f]=\limsup_{r\to+\infty}\frac{\exp(\alpha(\log\left(T_f\left(r\right)\right)))}{\left(\exp\left(\beta\left(\log(\gamma(r))\right)\right)\right)^{\rho_{(\alpha,\beta,\gamma)}[f]}}.$$

In this line, Biswas et al. [4] have recently introduced the definition of another growth indicator (α,β,γ) lower type of a meromorphic function f which is as follows:

Definition 1.5. [4] *The (α,β,γ) lower type denoted by $\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f having finite positive (α,β,γ) order $\left(0<\rho_{(\alpha,\beta,\gamma)}[f]<+\infty\right)$ are defined as :*

$$\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]=\liminf_{r\to+\infty}\frac{\exp(\alpha(\log\left(T_f\left(r\right)\right)))}{\left(\exp\left(\beta\left(\log(\gamma(r))\right)\right)\right)^{\rho_{(\alpha,\beta,\gamma)}[f]}}.$$

It is obvious that $0\leq\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]\leq\sigma_{(\alpha,\beta,\gamma)}[f]\leq+\infty$.

In this paper, we aim to establish some results depending on the comparative growth properties of composite transcendental entire or meromorphic functions and some special type of differential polynomials generated by one of the factors on the basis of (α,β,γ) type and (α,β,γ) lower type.

2. **Lemmas**

In this section, we present some lemmas which will be needed in the sequel.

Lemma 2.1. [3] *Let f be a transcendental meromorphic function and $F = f^n Q[f]$ where $Q[f]$ is a differential polynomial in f, then for any $n \geq 1$*

$$\begin{aligned} T_f(r) &= O\{T_F(r)\} \text{ as } r \to \infty \\ \text{and } T_F(r) &= O\{T_f(r)\} \text{ as } r \to \infty. \end{aligned}$$

Lemma 2.2. [5] *Let f be a transcendental meromorphic function and $F = f^n Q[f]$ where $Q[f]$ is a differential polynomial in f, then for any $n \geq 1$,*

$$\rho_{(\alpha,\beta,\gamma)}[F] = \rho_{(\alpha,\beta,\gamma)}[f] \text{ and } \lambda_{(\alpha,\beta,\gamma)}[F] = \lambda_{(\alpha,\beta,\gamma)}[f].$$

Lemma 2.3. *Let f be a transcendental meromorphic function and $F = f^n Q[f]$ where $Q[f]$ is a differential polynomial in f, then for any $n \geq 1$,*

$$\sigma_{(\alpha,\beta,\gamma)}[F] = \sigma_{(\alpha,\beta,\gamma)}[f] \text{ and } \overline{\sigma}_{(\alpha,\beta,\gamma)}[F] = \overline{\sigma}_{(\alpha,\beta,\gamma)}[f].$$

Proof. There exist some constants p, $q > 1$ such that for all sufficiently large r we have

$$T_f(r) < p \cdot T_F(r) \tag{1}$$

and

$$T_F(r) < q \cdot T_f(r). \tag{2}$$

Now from (1) it follows that for all sufficiently large values of r,

$$\begin{aligned} \log T_f(r) &< \log T_F(r) + \log p, \\ \text{which implis } \alpha(\log T_f(r)) &< (1+o(1))\alpha(\log T_F(r)), \\ \text{therefore } \exp[(\alpha(\log T_f(r))] &< \exp[(1+o(1))\,\alpha(\log T_F(r))]. \end{aligned}$$

Now using Lemma 2.2, we have for all sufficiently large values of r,

$$\frac{\exp[(\alpha(\log T_f(r))]}{(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}} < \frac{\exp[(1+o(1))\,\alpha(\log T_F(r))]}{(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[F]}},$$

which implies

$$\begin{aligned} \limsup_{r\to+\infty} \frac{\exp[(\alpha(\log T_f(r))]}{(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}} &< \limsup_{r\to+\infty} \frac{\exp[(1+o(1))\,\alpha(\log T_F(r))]}{(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[F]}}, \\ \text{hence, } \sigma_{(\alpha,\beta,\gamma)}[f] &\leq \sigma_{(\alpha,\beta,\gamma)}[F]. \end{aligned} \tag{3}$$

Similarly, from (2) we find that for all sufficiently large values of r,

$$\begin{aligned} \log T_F(r) &< \log T_f(r) + \log q, \\ \text{so, } \alpha(\log T_F(r)) &< (1+o(1))\alpha(\log T_f(r)), \\ \text{hence, } \exp\left[\alpha(\log T_F(r))\right] &< \exp\left[(1+o(1))\alpha(\log T_f(r))\right]. \end{aligned}$$

Now using Lemma 2.2, we have for all sufficiently large values of r,

$$\begin{aligned} \frac{\exp\left[\alpha(\log T_F(r))\right]}{\left(\exp\left(\beta\left(\log(\gamma(r))\right)\right)\right)^{\rho_{(\alpha,\beta,\gamma)}[F]}} &< \frac{\exp\left[(1+o(1))\alpha(\log T_f(r))\right]}{\left(\exp\left(\beta\left(\log(\gamma(r))\right)\right)\right)^{\rho_{(\alpha,\beta,\gamma)}[f]}}, \\ \text{so, } \limsup_{r\to+\infty}\frac{\exp\left[\alpha(\log T_F(r))\right]}{\left(\exp\left(\beta\left(\log(\gamma(r))\right)\right)\right)^{\rho_{(\alpha,\beta,\gamma)}[F]}} &< \limsup_{r\to+\infty}\frac{\exp\left[(1+o(1))\alpha(\log T_f(r))\right]}{\left(\exp\left(\beta\left(\log(\gamma(r))\right)\right)\right)^{\rho_{(\alpha,\beta,\gamma)}[f]}}, \\ \text{(4)} \qquad \text{hence, } \sigma_{(\alpha,\beta,\gamma)}[F] &\leq \sigma_{(\alpha,\beta,\gamma)}[f]. \end{aligned}$$

Therefore, from (3) and (4), we get that

$$\sigma_{(\alpha,\beta,\gamma)}[F] = \sigma_{(\alpha,\beta,\gamma)}[f].$$

In a similar manner, $\overline{\sigma}_{(\alpha,\beta,\gamma)}[F] = \overline{\sigma}_{(\alpha,\beta,\gamma)}[f]$.

Thus, the lemma follows. □

3. **Main results**

In this section, we present the main results of the paper.

Theorem 3.1. *Let f be a transcendental meromorphic function and g be an entire function such that $\rho_{(\alpha,\beta,\gamma)}[f\circ g] = \rho_{(\alpha,\beta,\gamma)}[f]$, $0 < \overline{\sigma}_{(\alpha,\beta,\gamma)}[f] \leq \sigma_{(\alpha,\beta,\gamma)}[f] < +\infty$ and $\overline{\sigma}_{(\alpha,\beta,\gamma)}[f\circ g] = +\infty$. Also, let $F = f^n Q[f]$ where $Q[f]$ is a differential polynomial in f for some $n \geq 1$, then*

$$\lim_{r\to+\infty}\frac{\exp(\alpha(\log(T_{f\circ g}(r))))}{\exp(\alpha(\log(T_F(r))))} = +\infty.$$

Proof. Suppose that the conclusion of the theorem does not hold. Then we can find a constant $\Delta > 0$ such that for a sequence of values of r tending to infinity

$$\exp(\alpha(\log(T_{f\circ g}(r)))) \leq \Delta\cdot\exp(\alpha(\log(T_F(r)))). \tag{5}$$

It follows from Lemma 2.3 and Definition 1.2 that for all sufficiently large values of r,

(6)

$$\exp(\alpha(\log(T_F(r)))) \leq \left(\sigma_{(\alpha,\beta,\gamma)}[f] + \varepsilon\right)\left(\exp(\beta(\log(\gamma(r))))\right)^{\rho_{(\alpha,\beta,\gamma)}[f]}.$$

From (5) and (6), for a sequence of values of r tending to $+\infty$, we have

$$\exp(\alpha(\log(T_{f\circ g}(r)))) \leq \Delta\left(\sigma_{(\alpha,\beta,\gamma)}[f]+\varepsilon\right)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]},$$

$$\text{which implies, } \frac{\exp(\alpha(\log(T_{f\circ g}(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f\circ g]}} \leq \Delta(\sigma_{(\alpha,\beta,\gamma)}[f]+\epsilon),$$

$$\text{therefore, } \liminf_{r\to+\infty}\frac{\exp(\alpha(\log(T_{f\circ g}(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f\circ g]}} < +\infty.$$

Hence from Definition 1.3, we have

$$\overline{\sigma}_{(\alpha,\beta,\gamma)}[f\circ g] < +\infty.$$

We obtain a contradiction.

Thus, the theorem follows. □

Remark 3.2. *Theorem 3.1 is also valid with "limit superior" instead of "limit" if "*$\lambda_{(\alpha,\beta,\gamma)}[f\circ g]=+\infty$*" is replaced by "*$\rho_{(\alpha,\beta,\gamma)}[f\circ g]=+\infty$*" and the other conditions remain the same.*

Theorem 3.3. *Let f be a transcendental meromorphic function and g be an entire function such that $0<\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]\leq\sigma_{(\alpha,\beta,\gamma)}[f]<+\infty$, $0<\overline{\sigma}_{(\alpha,\beta,\gamma)}[f\circ g]\leq\sigma_{(\alpha,\beta,\gamma)}[f\circ g]<+\infty$ and $\rho_{(\alpha,\beta,\gamma)}[f]=\rho_{(\alpha,\beta,\gamma)}[f\circ g]$. Also, let $F=f^nQ[f]$ where $Q[f]$ is a differential polynomial in f for some $n\geq 1$, then*

$$\begin{aligned}\frac{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f\circ g]}{\sigma_{(\alpha,\beta,\gamma)}[f]} &\leq \liminf_{r\to+\infty}\frac{\exp(\alpha(\log(T_{f\circ g}(r))))}{\exp(\alpha(\log(T_F(r))))}\\ &\leq \min\left\{\frac{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f\circ g]}{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]},\frac{\sigma_{(\alpha,\beta,\gamma)}[f\circ g]}{\sigma_{(\alpha,\beta,\gamma)}[f]}\right\}\\ &\leq \max\left\{\frac{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f\circ g]}{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]},\frac{\sigma_{(\alpha,\beta,\gamma)}[f\circ g]}{\sigma_{(\alpha,\beta,\gamma)}[f]}\right\}\\ &\leq \limsup_{r\to+\infty}\frac{\exp(\alpha(\log(T_{f\circ g}(r))))}{\exp(\alpha(\log(T_F(r))))} \leq \frac{\sigma_{(\alpha,\beta,\gamma)}[f\circ g]}{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]}.\end{aligned}$$

Proof. From the definitions of $\sigma_{(\alpha,\beta,\gamma)}[f]$, $\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]$, $\sigma_{(\alpha,\beta,\gamma)}[f\circ g]$ and $\overline{\sigma}_{(\alpha,\beta,\gamma)}[f\circ g]$ and using Lemma 2.2 and Lemma 2.3, we have for arbitrary positive $\varepsilon(>0)$ and for all sufficiently large values of r that

(7)

$$\exp(\alpha(\log(T_{f\circ g}(r)))) \leq \left(\sigma_{(\alpha,\beta,\gamma)}[f\circ g]+\varepsilon\right)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f\circ g]},$$

(8)

$$\exp(\alpha(\log(T_{f\circ g}(r)))) \geq \left(\overline{\sigma}_{(\alpha,\beta,\gamma)}[f\circ g]-\varepsilon\right)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f\circ g]},$$

(9)
$$\exp(\alpha(\log(T_F(r)))) \leq \left(\sigma_{(\alpha,\beta,\gamma)}[f]+\varepsilon\right)\left(\exp(\beta(\log(\gamma(r))))\right)^{\rho_{(\alpha,\beta,\gamma)}[f]},$$

(10)
$$\exp(\alpha(\log(T_F(r)))) \geq \left(\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]-\varepsilon\right)\left(\exp(\beta(\log(\gamma(r))))\right)^{\rho_{(\alpha,\beta,\gamma)}[f]}.$$

For a sequence of values of r tending to infinity, we get that

(11)
$$\exp(\alpha(\log(T_{f\circ g}(r)))) \geqslant (\sigma_{(\alpha,\beta,\gamma)}[f\circ g]-\varepsilon)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f\circ g]},$$

(12)
$$\exp(\alpha(\log(T_{f\circ g}(r)))) \leq \left(\overline{\sigma}_{(\alpha,\beta,\gamma)}[f\circ g]+\varepsilon\right)\left(\exp(\beta(\log(\gamma(r))))\right)^{\rho_{(\alpha,\beta,\gamma)}[f\circ g]},$$

(13)
$$\exp(\alpha(\log(T_F(r)))) \leq \left(\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]+\varepsilon\right)\left(\exp(\beta(\log(\gamma(r))))\right)^{\rho_{(\alpha,\beta,\gamma)}[f]},$$

(14)
$$\exp(\alpha(\log(T_F(r)))) \geq \left(\sigma_{(\alpha,\beta,\gamma)}[f]-\varepsilon\right)\left(\exp(\beta(\log(\gamma(r))))\right)^{\rho_{(\alpha,\beta,\gamma)}[f]}.$$

Now, from (8), (9) and the condition $\rho_{(\alpha,\beta,\gamma)}[f\circ g]=\rho_{(\alpha,\beta,\gamma)}[f]$, it follows for all sufficiently large values of r that

$$\frac{\exp(\alpha(\log(T_{f\circ g}(r))))}{\exp(\alpha(\log(T_F(r))))} \geqslant \frac{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f\circ g]-\varepsilon}{\sigma_{(\alpha,\beta,\gamma)}[f]+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain from the above that

$$\liminf_{r\to+\infty}\frac{\exp(\alpha(\log(T_{f\circ g}(r))))}{\exp(\alpha(\log(T_F(r))))} \geqslant \frac{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f\circ g]}{\sigma_{(\alpha,\beta,\gamma)}[f]}. \tag{15}$$

Combining (12) and (10) and the condition $\rho_{(\alpha,\beta,\gamma)}[f\circ g]=\rho_{(\alpha,\beta,\gamma)}[f]$, we get for a sequence of values of r tending to infinity that

$$\frac{\exp(\alpha(\log(T_{f\circ g}(r))))}{\exp(\alpha(\log(T_F(r))))} \leq \frac{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f\circ g]+\varepsilon}{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from the above that

$$\liminf_{r\to+\infty}\frac{\exp(\alpha(\log(T_{f\circ g}(r))))}{\exp(\alpha(\log(T_F(r))))} \leq \frac{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f\circ g]}{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]}. \tag{16}$$

Now, from (8), (13) and the condition $\rho_{(\alpha,\beta,\gamma)}[f\circ g]=\rho_{(\alpha,\beta,\gamma)}[f]$, we obtain for a sequence of values of r tending to infinity that

$$\frac{\exp(\alpha(\log(T_{f\circ g}(r))))}{\exp(\alpha(\log(T_F(r))))} \geq \frac{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f\circ g]-\varepsilon}{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to+\infty}\frac{\exp(\alpha(\log(T_{f\circ g}(r))))}{\exp(\alpha(\log(T_F(r))))}\geq\frac{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f\circ g]}{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]}. \tag{17}$$

In view of the condition $\rho_{(\alpha,\beta,\gamma)}[f\circ g]=\rho_{(\alpha,\beta,\gamma)}[f]$, it follows from (10) and (7), for all sufficiently large values of r that

$$\frac{\exp(\alpha(\log(T_{f\circ g}(r))))}{\exp(\alpha(\log(T_F(r))))}\leq\frac{\sigma_{(\alpha,\beta,\gamma)}[f\circ g]+\varepsilon}{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to+\infty}\frac{\exp(\alpha(\log(T_{f\circ g}(r))))}{\exp(\alpha(\log(T_F(r))))}\leq\frac{\sigma_{(\alpha,\beta,\gamma)}[f\circ g]}{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]}. \tag{18}$$

Now, from (7), (14) and the condition $\rho_{(\alpha,\beta,\gamma)}[f\circ g]=\rho_{(\alpha,\beta,\gamma)}[f]$, it follows that, for a sequence of values of r tending to infinity

$$\frac{\exp(\alpha(\log(T_{f\circ g}(r))))}{\exp(\alpha(\log(T_F(r))))}\leq\frac{\sigma_{(\alpha,\beta,\gamma)}[f\circ g]+\varepsilon}{\sigma_{(\alpha,\beta,\gamma)}[f]-\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\exp(\alpha(\log(T_{f\circ g}(r))))}{\exp(\alpha(\log(T_F(r))))}\leq\frac{\sigma_{(\alpha,\beta,\gamma)}[f\circ g]}{\sigma_{(\alpha,\beta,\gamma)}[f]}. \tag{19}$$

So, combining (9) and (11) and in view of the condition $\rho_{(\alpha,\beta,\gamma)}[f\circ g]$ $=\rho_{(\alpha,\beta,\gamma)}[f]$, we get for a sequence of values of r tending to infinity that

$$\frac{\exp(\alpha(\log(T_{f\circ g}(r))))}{\exp(\alpha(\log(T_F(r))))}\geqslant\frac{\sigma_{(\alpha,\beta,\gamma)}[f\circ g]-\varepsilon}{\sigma_{(\alpha,\beta,\gamma)}[f]+\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to+\infty}\frac{\exp(\alpha(\log(T_{f\circ g}(r))))}{\exp(\alpha(\log(T_F(r))))}\geqslant\frac{\sigma_{(\alpha,\beta,\gamma)}[f\circ g]}{\sigma_{(\alpha,\beta,\gamma)}[f]}. \tag{20}$$

Thus the claim of the theorem follows from (15) , (16) , (17), (18) , (19) and (20) . □

The proof of the following result is similar to that of Theorem 3.3 and therefore we omit its proof.

Theorem 3.4. *Le Let f be a transcendental meromorphic function and g be a transcendental entire function such that $0<\overline{\sigma}_{(\alpha,\beta,\gamma)}[g]\leq\sigma_{(\alpha,\beta,\gamma)}[g]<+\infty$ and $0<\overline{\sigma}_{(\alpha,\beta,\gamma)}[f\circ g]\leq\sigma_{(\alpha,\beta,\gamma)}[f\circ g]<+\infty$ and*

$\rho_{(\alpha,\beta,\gamma)}[g] = \rho_{(\alpha,\beta,\gamma)}[f \circ g]$. *Also, let* $G = g^m Q[g]$ *where* $Q[g]$ *is a differential polynomial in* g. *Then for some* $m \geq 1$,

$$\begin{aligned}
\frac{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f \circ g]}{\sigma_{(\alpha,\beta,\gamma)}[g]} &\leq \liminf_{r \to +\infty} \frac{\exp(\alpha(\log (T_{f \circ g}(r))))}{\exp(\alpha(\log (T_G(r))))} \\
&\leq \min \left\{ \frac{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f \circ g]}{\overline{\sigma}_{(\alpha,\beta,\gamma)}[g]}, \frac{\sigma_{(\alpha,\beta,\gamma)}[f \circ g]}{\sigma_{(\alpha,\beta,\gamma)}[g]} \right\} \\
&\leq \max \left\{ \frac{\overline{\sigma}_{(\alpha,\beta,\gamma)}[f \circ g]}{\overline{\sigma}_{(\alpha,\beta,\gamma)}[g]}, \frac{\sigma_{(\alpha,\beta,\gamma)}[f \circ g]}{\sigma_{(\alpha,\beta,\gamma)}[g]} \right\} \\
&\leq \limsup_{r \to +\infty} \frac{\exp(\alpha(\log (T_{f \circ g}(r))))}{\exp(\alpha(\log (T_G(r))))} \leq \frac{\sigma_{(\alpha,\beta,\gamma)}[f \circ g]}{\overline{\sigma}_{(\alpha,\beta,\gamma)}[g]}.
\end{aligned}$$

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