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ON m-TOPOLOGY AND u-TOPOLOGY OF SUPERTOPOLOGICAL RINGS

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Abstract. The aim of the paper is to identify the largest supertopological subrings, endowed with the uniform topology or with the m-topology, of the ring of real functions defined on a Tychonoff space X. Also, some special cases for the space X are investigated.

1. Introduction and Preliminaries

The theory of topological rings is studied in detail in [1, 6, 7, 10]. For a Tychonoff space X, we denote by $\mathbb{R}^X$ the ring of all real valued functions defined on X. Two of the important subrings $\mathbb{R}^X$ of are $C(X)$, the set of all continuous functions in $\mathbb{R}^X$, and $C^*(X)$, the set of all bounded and continuous members of $\mathbb{R}^X$. The algebraic properties of the rings $C(X)$ and $C^*(X)$ and topological properties of X have been studied extensively in the literature.

Besides studying the algebraic properties of $C(X)$, one can also define many interesting topologies on $C(X)$. Consequently, one may study the interaction between various algebraic structures of $C(X)$ with the corresponding topology on it.

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Two commonly studied topologies on $C(X)$ and $C^*(X)$ are the u-topology (uniform topology) and the m-topology. Both of these topologies can be defined on $\mathbb{R}^X$. In many aspects, the u-topology is the most relevant for studying $C^*(X)$, while the m-topology is appropriate to study the ring $C(X)$. It is known that $C^*(X)$ equipped with the u-topology forms a topological ring while $C(X)$ need not. However, $C(X)$ equipped with the m-topology is a topological ring. But $\mathbb{R}^X$ with any of these topologies does not form a topological ring in general. There are several interesting subrings that are intermediate between $C^*(X)$ and $\mathbb{R}^X$, such as the ring of all Baire one functions and the ring of all locally bounded functions.

Kohli and Singh defined D-supercontinuity and studied the properties of D-supercontinuos functions in [8]. We use supertopological rings introduced in [9] by B. Vashishth and D. Singh.

We begin with definitions and remarks which will be used throughout the article.

Definition 1 ([8])**.** *A function $f\colon X \to Y$ from topological space X to topological space Y is said to be* D-supercontinuous *if for each $x \in X$ and each open set $U \subseteq Y$ containing $f(x)$ there exists an open F_σ-set $V \subseteq X$ containing x such that $f(V) \subseteq U$.*

For more theory and equivalent definitions of D-supercontinuous functions, see [8].

Definition 2. *A topological space G that is also a group, with the operation $\cdot$, where $G\times G$ carries product topology, is a* supertopological group *if the mappings*

$$g_1\colon G \times G \to G, \quad (x,y) \mapsto xy, \qquad \textit{and} \qquad g_2\colon G \to G, \quad x \mapsto x^{-1}$$

are D-supercontinuous.

Definition 3. *A topological space A that is also a ring with operations $+$ and $\cdot$ where $A \times A$ carries product topology, is a* supertopological ring *if the mappings*

$$\begin{aligned} a_1\colon A \times A &\to A,\ a_1(x,y) = x + y, \\ a_2\colon A &\to A,\ a_2(x) = -x \\ &\textit{and} \\ a_3\colon A \times A &\to A,\ a_3(x,y) = x \cdot y \end{aligned}$$

are D-supercontinuous.

Definition 4 ([8])**.** *A set U in a topological space X is said to be* d-open *if for each $x \in U$, there exists an open F_σ-set H such that $x \in H \subseteq U$. Complement of a d-open set is called* d-closed.

Throughout this paper a d-neighborhood of a point x will mean an open F_σ-set containing x.

2. m-topology and u-topology

Throughout this section X is assumed to be a Tychonoff space. We first define the notions of m-topology and u-topology.

Definition 5. *The u-topology or uniform topology is defined on $\mathbb{R}^X$ by taking all sets of the form*

$$B_u(f, \varepsilon) = \{g \in \mathbb{R}^X : \ |f(x) - g(x)| < \varepsilon \text{ for all } x \in X\}$$

as a base for the neighbourhood system at $f \in \mathbb{R}^X$.

Definition 6. *The m-topology is defined on $\mathbb{R}^X$ by taking all sets of the form*

$$B_m(f, \eta) = \{g \in \mathbb{R}^X : \ |f(x) - g(x)| < \eta(x) \text{ for all } x \in X\} \quad (\eta \in U_+(X))$$

as a base for the neighbourhood system at $f \in \mathbb{R}^X$. Here $U_+(X)$ denotes the set of all positive units of $C(X)$.

Theorem 7. *Let τ be the topology on $\mathbb{R}^X$ determined by taking all sets of the form*

$$B_u(f, \gamma) = \{g \in \mathbb{R}^X : \ |f(x) - g(x)| < \gamma(x) \text{ for all } x \in X\} \quad (\gamma \in U_+^*(X))$$

as a base for the neighborhood system at f for each $f \in \mathbb{R}^X$, where $U_+^*(X)$ denotes the set of all positive units in $C^*(X)$. Then $\tau = \tau_u$ on $\mathbb{R}^X$.

There is a considerable difference between the units of the rings $C(X)$ and $C^*(X)$. A function $f \in C(X)$ is a positive unit of $C(X)$ if and only if $f(x) > 0$. But a function $f \in C^*(X)$ is a positive unit of $C^*(X)$ if and only if $f(x) > 0$ and $\frac{1}{f} \in C^*(X)$. Equivalently, $f \in C^*(X)$ is a positive unit of $C^*(X)$ if and only if $\inf\{f(x) : x \in X\} > 0$. So it is clear that every positive unit of $C^*(X)$ is also a positive unit of $C(X)$, but the converse need not be true.

To identify the largest subrings of $\mathbb{R}^X$, which are topological rings endowed with τ_u and τ_m, respectively, we define two families of functions as follows:

$$B(X) = \{f \in \mathbb{R}^X : \text{there exists } \xi \in U_+^*(X) \text{ s.t. } |f(x)| < \xi(x) \text{ for all } x \in X\},$$

$$D(X) = \{f \in \mathbb{R}^X : \text{there exists } \varphi \in U_+(X) \text{ s.t. } |f(x)| < \varphi(x) \text{ for all } x \in X\}.$$

Note that $B(X)$ and $D(X)$ are subrings of $\mathbb{R}^X$ and $B(X)$ is same as the family of all bounded functions in $\mathbb{R}^X$. Clearly, $C(X) \subseteq D(X)$ and $B(X) \subseteq D(X)$. Also $D(X) = B(X)$ if X is pseudocompact.

Clearly, $D(X) \subset LB(X)$ where we used the notation

$$LB(X) = \{\, f \in \mathbb{R}^X : f \text{ is locally bounded on } X\,\}.$$

A function $f : X \to \mathbb{R}$ is said to be *locally bounded* if for each $x \in X$ there exists a neighbourhood U of x and a constant $M > 0$ such that

$$|f(y)| \leq M \quad \text{for all } y \in U.$$

Definition 8 ([4])**.** *A space X is called a* cb-space *if for each $g \in LB(X)$, there exists $f \in C(X)$ such that $|g| \leq f$.*

Theorem 9. *$D(X) = LB(X)$ if X is a cb-space.*

Proof. Since $D(X) \subseteq LB(X)$ always holds, it suffices to show $LB(X) \subseteq D(X)$. Let $g \in LB(X)$. As X is a cb-space, there exists $f \in C(X)$ such that

$$|g| \leq f.$$

Define $\varphi = f + 1$. Then $\varphi \in U_+(X)$ and for all $x \in X$,

$$|g(x)| \leq f(x) < \varphi(x).$$

Hence $g \in D(X)$, and therefore $LB(X) \subseteq D(X)$. This proves

$$D(X) = LB(X).$$

∎

It is known that a *cb*-space is countably paracompact, and that a normal space is a *cb*-space if and only if it is countably paracompact. Consequently, for a space X which is not countably paracompact , $D(X) \neq LB(X)$.

Theorem 10. *$D(X)$ endowed with τ_m is a supertopological ring.*

Proof. The supercontinuity of $(f,g)\mapsto f+g$ is easy to check.

We prove the supercontinuity of $(f,g)\mapsto fg$.

Let $f,g\in D(X)$. So, there exists $\varphi_f,\varphi_g\in U_+(X)$ such that $|f(x)|<\varphi_f(x)$ and $|g(x)|<\varphi_g(x)$ for all $x\in X$.

Let $B_m(fg,\eta)$ be any basic d-neighbourhood of fg in $(D(X),\tau_m)$ for some $\eta\in U_+(X)$. Consider the basic d-neighbourhoods $B_m(f,\eta_1)$ and $B_m(g,\eta_2)$ of f and g, respectively, for

$$\eta_1=\frac{\eta}{2(1+\varphi_g)},\qquad \eta_2=\frac{\eta}{2(1+\varphi_f+\eta_1)}.$$

It is enough to show that for any $h_1\in B_m(f,\eta_1)$ and $h_2\in B_m(g,\eta_2)$ we have $h_1h_2\in B_m(fg,\eta)$.

This follows, since for all $x\in X$

$$\begin{aligned}|(fg)(x)-(h_1h_2)(x)| &\le |g(x)|\,|f(x)-h_1(x)|+|h_1(x)|\,|g(x)-h_2(x)|\\ &<\varphi_g(x)\eta_1(x)+|h_1(x)|\,\eta_2(x)\\ &<\eta(x).\end{aligned}$$

∎

Theorem 11. *Let $S(X)$ be a subring of $\mathbb{R}^X$. The following are equivalent:*

(1) *$S(X)$ endowed with τ_m is a supertopological ring.*
(2) *$S(X)\subseteq D(X)$.*

Proof. (2) $\Rightarrow$ (1) follows from Theorem 2.6. For (1) $\Rightarrow$ (2) we suppose that $S(X)$ is not contained in $D(X)$. We show that the pointwise multiplication $(f,g)\mapsto fg$ is not supercontinuous at point $(0_X,f)$ where 0_X is the constant function zero on X.

Consider the basic d-neighbourhood $B_m(0_X,1)$ of the function $0_Xf=0_X$ in $(S(X),\tau_m)$. Since $f\notin D(X)$ for every $\eta\in U_+(X)$ there exists a point $x_\eta\in X$ such that $|f(x_\eta)|\ge\frac{2}{\eta(x_\eta)}$, that is, $\left|\frac{\eta(x_\eta)}{2}f(x_\eta)\right|\ge 1$.

Therefore for any $\eta,\mu\in U_+(X)$, we have $\frac{\eta}{2}\in B_m(0_X,\eta)$ and $f\in B_m(f,\mu)$, but $\frac{\eta}{2}f\notin B_m(0_X,1)$. ∎

Corollary 12. *$LB(X)$ equipped with τ_m is a supertopological ring if X is a cb-space.*

Proof. Since X is a *cb*-space, every locally bounded real-valued function on X is dominated by a strictly positive continuous function. Hence

$$LB(X) \subseteq D(X).$$

Applying Theorem 11 with $S(X) = LB(X)$, τ_m is a supertopological ring. ∎

Theorem 13. *For a first countable space the following are equivalent:*

(1) X *is discrete.*
(2) $\mathbb{R}^X = D(X)$.
(3) $\mathbb{R}^X$ *endowed with* τ_m *is a supertopological ring.*
(4) $\mathbb{R}^X = LB(X)$.

Proof. (1) ⇒ (2) is immediate and (2) ⇒ (3) ⇒ (4) follows from Theorem 2.7. For (4) ⇒ (1) we see that if we suppose there is a non isolated point $x_0 \in X$, since X is first countable, there exists a sequence (x_n) of distinct points in $X \setminus \{x_0\}$ which converges to x_0. Define a function $f\colon X \to \mathbb{R}$ such that $f(x_n) = n$ for every $n \in \mathbb{N}$ and $f(x) = 0$ for all $x \in X \setminus \{x_n : n \in \mathbb{N}\}$. It is easy to see that $f \notin LB(X)$. Hence we arrive at a contradiction. ∎

Theorem 14. $B(X)$ *is the largest subring of* $\mathbb{R}^X$ *which is a supertopological ring endowed with* τ_u.

Proof. The proof is similar to the proofs of Theorem 10 and Theorem 11. First, one shows that $B(X)$ endowed with τ_u is a supertopological ring. Since every $f \in B(X)$ is bounded by a constant, uniform neighbourhoods control products exactly as in Theorem 10, and hence both addition and multiplication are supercontinuous with respect to τ_u.

Next, let $S(X)$ be any subring of $\mathbb{R}^X$ which is a supertopological ring under τ_u. If $S(X) \nsubseteq B(X)$, then there exists $f \in S(X)$ which is unbounded. Using basic τ_u-neighbourhoods of the zero function, one checks that pointwise multiplication fails to be supercontinuous at $(0_X, f)$, contradicting the assumption.

Therefore $S(X) \subseteq B(X)$, and hence $B(X)$ is the largest subring of $\mathbb{R}^X$ which is a supertopological ring with respect to τ_u. ∎

Corollary 15. $\mathbb{R}^X$ *equipped with* τ_u *is a supertopological ring if and only if* X *is finite.*

Corollary 16. *Every subring of $\mathbb{R}^X$ which forms a supertopological ring under τ_u is also a supertopological ring under τ_m.*

Corollary 17. *For a space X the following are equivalent:*

(1) *$LB(X)$ endowed with τ_u is a supertopological ring;*
(2) *$B(X) = LB(X)$;*
(3) *X is a pseudocompact cb space;*
(4) *X is countably compact.*

Proof. (1) $\Leftrightarrow$ (2) follows from Theorem 2.10. (2) $\Leftrightarrow$ (3) follows from the fact $D(X) = B(X)$ if and only if X is pseudocompact. ∎

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